Ruin Probabilities

The *gambler's ruin* problem is one of the oldest and most celebrated problems in probability theory, with roots going back to the correspondence between Pascal and Fermat in the seventeenth century. In its simplest form, a gambler bets repeatedly on fair coin flips and faces two questions: what is the probability of accumulating A dollars before losing B dollars, and how long should she expect to play before one outcome or the other occurs? Despite the elementary setup, these questions lead naturally to some of the most powerful tools in probability: martingales, harmonic functions, and the theory of stopping times.

In these notes, which follow Steele (2001) closely, we analyze both questions in the setting of a simple symmetric random walk. We derive closed-form answers using second-order linear difference equations — a technique that recurs throughout stochastic processes and mathematical finance. The analysis reveals a striking result: starting from zero, the probability of winning A dollars before losing B dollars is exactly $B/(A+B)$, and the expected duration of the game is AB. Both answers carry an elegant symmetry and intuition well beyond the gambling context, appearing naturally in models of sequential testing, insurance risk, and the pricing of barrier options.

A Simple Random Walk

A simple *random walk* models the wealth of a gambler betting on fair coin flips. To start, let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent random variables, each taking values 1 or -1 with equal probability $\frac{1}{2}$. Therefore, at each turn the gambler can either win one dollar or lose one dollar.

By construction, the sequence $\{X_i\}$ is iid with mean $\mathsf{E}(X_i) = 0.5 \times 1 + 0.5(-1) = 0$ and variance $\mathsf{V}(X_i) = 0.5 \times (1-0)^2 + 0.5 \times (-1-0)^2 = 1$.

Let S_0 be the initial wealth of the gambler. For $n \geq 1$, define

$$S_n = S_0 + \sum_{i=1}^{n} X_i.$$

We will assume that the wealth may become negative, which requires the gambler to honor any potential losses.

Define τ as the first time $n > 0$ when S_n reaches either A or $-B$:

$$\tau = \min\{n > 0 : S_n = A \text{ or } S_n = -B\}.$$

At the random time τ, S_τ is either A or $-B$, and we seek to compute the probability of the gambler winning A dollars before losing B dollars, that is $P(S_\tau = A \mid S_0 = 0)$. Also, it is almost natural to ask how long do we need to wait for the gambler to either win A dollars or to lose B dollars. We will denote this expectation by $E(\tau \mid S_0 = 0)$.

Behavior of τ

To solve these problems, we need first to make sure that the questions have an answer! That is, how do we know for sure that the gambler's net winnings will eventually reach A or $-B$?

Clearly, if the gambler wins $A + B$ times in a row then she is guaranteed to reach a wealth level of A even if she starts at $S_0 = -B + 1$, the worst case starting point. Note that if the gambler starts at $S_0 = -B$ then the game is already over. Let's denote by E_k the event that the gambler wins consecutively in steps $k(A + B) + 1$ through $(k + 1)(A + B)$. Note that if we take $k = 0$, $A = 3$ and $B = 2$, then $A + B = 5$. If you win all 5 steps in the block, the game is guaranteed to end within the block — from $S_0 = -1$, just 4 wins suffice to reach $A = 3$, so 5 consecutive wins is more than enough.

Clearly, the events $\{E_k\}$ are independent and

$$p = P(E_k) = \left(\frac{1}{2}\right)^{A+B} > 0.$$

Since each E_k guarantees the game ends by step $(k + 1)(A + B)$, the event $\{\tau \geq n(A + B)\}$ implies that none of $E_0, \ldots, E_{n-1}$ occurred. Therefore,

$$P(\tau \geq n(A + B)) \leq P\left(\cap_{k=0}^{n-1} E_k^C\right) = (1 - p)^n \xrightarrow[n \to \infty]{} 0.$$

Since we also have that $P(\tau = \infty) \leq P(\tau \geq n(A + B))$, for all $n \geq 0$, we see that $P(\tau = \infty) = 0$. In words, it must take a finite time for the wealth of the gambler to reach A or $-B$.

Furthermore, for $d = 1, 2, \ldots$ we also have that for $k \geq 1$

$$\tau^d \mathbf{1}_{\{(k-1)(A+B) < \tau \leq k(A+B)\}} \leq k^d (A + B)^d \mathbf{1}_{\{(k-1)(A+B) < \tau\}}.$$

Summing over $k = 1, 2, \ldots$ we find that

$$\tau^d \leq \sum_{k=1}^{\infty} k^d (A + B)^d \mathbf{1}_{\{(k-1)(A+B) < \tau\}}.$$

Finally, taking expectations on both sides yields

$$E(\tau^d) \leq \sum_{k=1}^{\infty} k^d (A + B)^d \, P((k - 1)(A + B) < \tau) \leq \sum_{k=1}^{\infty} k^d (A + B)^d (1 - p)^{k-1}.$$

The series on the right converges since

$$\lim_{k \to \infty} \frac{(k + 1)^d (A + B)^d (1 - p)^k}{k^d (A + B)^d (1 - p)^{k-1}} = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^d (1 - p) = 1 - p < 1.$$

Therefore, we have shown that $E(\tau^d) < \infty$ for any integer $d \geq 1$. We compile the results in the following result.

> **Stopping Time of the Random Walk**
>
> Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent random variables, each taking values 1 or -1

with equal probability $\frac{1}{2}$. For $n \geq 1$, define

$$S_n = S_0 + \sum_{i=1}^{n} X_i,$$

where $-B < S_0 < A$, and A and B are two positive integers. Define τ as the first time $n > 0$ when S_n reaches either A or $-B$:

$$\tau = \min\{n > 0 : S_n = A \text{ or } S_n = -B\}.$$

We then have that $P(\tau = \infty) = 0$ and $E(\tau^d) < \infty$ for any integer $d \geq 1$.

Solving for the Probability

In this section we solve for $P(S_\tau = A \mid S_0 = 0)$. To do so, we solve the more general problem for all starting values K. Since the gambler can keep playing without any explicit time limit, the problem depends only on the current wealth level rather than on time. In other words, the only relevant state variable is the wealth level.

Let's write

$$f(K) = P(S_\tau = A \mid S_0 = K).$$

In words, $f(K)$ denotes the probability of reaching A before $-B$ if we start with a wealth level $-B \leq K \leq A$. Clearly, $f(-B) = 0$ since if we start at $S_0 = -B$ the game is over. Also, $f(A) = 1$ since if we start at $S_0 = A$ we have reached our objective. These are our boundary conditions that will allow us to solve for $f(K)$.

Now, we need to get an equation. We note that if we start with $S_0 = K$ and we win, then our wealth increases to $K + 1$ and we start all over as if time is back at zero but our initial wealth is $K + 1$. Similarly, if we start with $S_0 = K$ and we lose, then our wealth decreases to $K - 1$ and the problem is the same with an initial wealth of $K - 1$. Thus, it must be the case that

$$f(K) = \frac{1}{2}f(K - 1) + \frac{1}{2}f(K + 1). \tag{1}$$

Equation (1) is a second-order linear difference equation in the wealth level K. Equations like this abound in probability and finance theory, as they capture the dynamics of quantities like probabilities, expectations or values. Solutions to equations like (1) are typically of the form $f(K) = z^K$ where z is a real or complex number. If we try with $f(K) = z^K$ in equation (1) we find

$$z^K = \frac{1}{2}z^{K-1} + \frac{1}{2}z^{K+1},$$

so that

$$z^2 - 2z + 1 = (z - 1)^2 = 0.$$

The only solution to the second order polynomial is 1, so we know that a constant is a solution. But there should be two linearly independent solutions, as is expected of a second-order linear difference equation. When there is a repeated root $z = 1$, a second independent solution is $K \cdot 1^K = K$, which we can verify directly: substituting $f(K) = K$ into (1) gives $K = \frac{1}{2}(K - 1) + \frac{1}{2}(K + 1)$, which holds trivially. The general solution to equation (1) is then

$$f(K) = \alpha + \beta K. \tag{2}$$

To solve for α and β in equation (2) we just use the fact that $f(-B) = 0$ and $f(A) = 1$, so that

$$\alpha - \beta B = 0,$$
$$\alpha + \beta A = 1.$$

This implies that $\alpha = \frac{B}{A+B}$ and $\beta = \frac{1}{A+B}$ so that

$$f(K) = \frac{B + K}{A + B}.$$

We can finally conclude that

$$P(S_\tau = A \mid S_0 = 0) = f(0) = \frac{B}{A + B}.$$

Therefore, the probability of making \$10 before losing \$10 is just $10/20 = 1/2$, and the probability of making \$10 before losing \$5 is $5/15 = 1/3$.

Solving for the Expectation

We now want to solve for how long does it take on average to either win \$A or lose \$B. That is, we want to compute $E(\tau \mid S_0 = 0)$. For this, like before we define

$$g(K) = E(\tau \mid S_0 = K),$$

and note that $g(-B) = g(A) = 0$, since the game has already ended at the boundary, so the expected additional time to stop is zero. We can then work the recursion as follows. If we are currently at $S_0 = K$, and we play once more, the expected time to reach the target or lose it all has increased by one. If we win it's the same problem with either one extra or one less unit of wealth, each happening with probability $1/2$. Thus,

$$g(K) = 1 + \frac{1}{2}g(K-1) + \frac{1}{2}g(K+1). \tag{3}$$

To solve for (3), we already have the homogeneous solution computed earlier. We can find a particular solution by trying γK^2 which is the next power of K. Therefore,

$$\gamma K^2 = 1 + \frac{\gamma}{2}(K-1)^2 + \frac{\gamma}{2}(K+1)^2,$$

or

$$K^2 = \frac{1}{\gamma} + \frac{1}{2}(K^2 - 2K + 1) + \frac{1}{2}(K^2 + 2K + 1).$$

Thus, we find that $1/\gamma + 1 = 0$ or $\gamma = -1$. The general solution to (3) is then

$$g(K) = \alpha + \beta K - K^2. \tag{4}$$

We determine the constants α and β in (4) using

$$\alpha - \beta B - B^2 = 0,$$
$$\alpha + \beta A - A^2 = 0.$$

We find that

$$\beta(A+B) = A^2 - B^2 = (A+B)(A-B),$$

so $\beta = A - B$ and $\alpha = (A - B)B + B^2 = AB$.

Thus, we conclude that

$$g(K) = AB + (A - B)K - K^2,$$

and

$$\mathsf{E}(\tau \mid S_0 = 0) = g(0) = AB.$$

References

Steele, J. Michael. 2001. *Stochastic Calculus and Financial Applications*. Vol. 45. Applications of Mathematics. Springer.