

Derivatives Pricing in Continuous Time

Price Processes

Let's start considering a non-dividend paying stock S over the time interval $[0, T]$. Of course, a stock that does not pay dividends forever is a bubble but we will focus on a finite period of time in which the stock does not pay dividends. The stock price process follows a geometric Brownian of the form

$$\frac{dS}{S} = \mu dt + \sigma dB,$$

and there is a money-market account β that grows at the risk-free rate r such that

$$\frac{d\beta}{\beta} = r dt.$$

Since our objective is to price derivatives written on S with payoffs given by some function of the stock price at time T , we can write the discount factor as

$$\frac{d\Lambda}{\Lambda} = -r dt - \lambda dB.$$

The stochastic part of the discount factor that matters for this application is the one that is perfectly correlated with the stock price process. For the moment, we assume that μ , r , σ and λ are all adapted process to the filtration on which all Brownian motions are adapted.

The pricing equation implies that λ should be the instantaneous Sharpe ratio of the stock price so that

$$\lambda = \frac{\mu - r}{\sigma}.$$

The Risk-Neutral Measure

Girsanov's theorem allows us to create Brownian motions under a different measure by using strictly positive martingales. The risk-neutral measure is a particular measure created by using the process $\mathcal{E} = \Lambda\beta$. Ito's lemma implies that

$$\frac{d\mathcal{E}}{\mathcal{E}} = \frac{d\Lambda}{\Lambda} + \frac{d\beta}{\beta} = -\lambda dB.$$

We can then compute

$$\mathcal{E}_t = \mathcal{E}_0 e^{-\int_0^t \lambda_s dB_s},$$

which is strictly positive as long $\mathcal{E}_0 > 0$. Note that many authors normalize $\Lambda_0 = 1$ and $\beta_0 = 1$ so that $\mathcal{E}_0 = 1$.

The previous expression implies that $\{\mathcal{E}_t\}$ is a strictly positive local martingale. If $\{\mathcal{E}_t\}$ is actually a martingale, we can create a new measure P^* such that

$$\frac{dP^*}{dP} = \frac{\mathcal{E}_T}{\mathcal{E}_0}.$$

Thus, for any \mathcal{F}_t -adapted process $\{X_t\}$ we have that

$$E^*(X_T) = E\left(\frac{dP^*}{dP} X_T\right) = E\left(\frac{\mathcal{E}_T}{\mathcal{E}_0} X_T\right).$$

There are many models in which $\{\mathcal{E}_t\}$ is a proper martingale. For example, if λ is constant, then it is not hard to show that $\{\mathcal{E}_t\}$ is a martingale. In the following, we assume that $\{\lambda_t\}$ is such that $\{\mathcal{E}_t\}$ is a strictly positive martingale.

Girsanov's theorem then implies that for $0 \leq t \leq T$ we have that

$$B_t^* = B_t - \int_0^t \left(\frac{d\mathcal{E}_s}{\mathcal{E}_s}\right) (dB_s)$$

is a P^* -Brownian motion. If $\mathcal{E} = \Lambda\beta$, we can compute

$$\left(\frac{d\mathcal{E}}{\mathcal{E}}\right)(dB) = \left(\frac{d\Lambda}{\Lambda} + \frac{d\beta}{\beta}\right)(dB) = -\lambda dt.$$

Thus, we can write the new Brownian motion in differential form:

$$dB^* = dB + \lambda dt.$$

The dynamics of S under P^* are then given by

$$\frac{dS}{S} = \mu dt + \sigma(dB^* - \lambda dt) = (\mu - \lambda\sigma) dt + \sigma dB^*. \quad (1)$$

Thus, we have that

$$\frac{dS}{S} = r dt + \sigma dB^*.$$

The previous expression implies that the drift of the stock is just the risk-free rate under P^* .

Consider now another asset V exposed to the same Brownian motion B ,

$$\frac{dV}{V} = \mu_V dt + \sigma_V dB. \quad (2)$$

It must also be the case that

$$\lambda = \frac{\mu - r}{\sigma} = \frac{\mu_V - r}{\sigma_V},$$

so that

$$\frac{dV}{V} = r dt + \sigma dB^*. \quad (3)$$

Thus, all assets under P^* earn the same rate of return equal to the risk-free rate. This is why we call the measure P^* the **risk-neutral measure**. In a risk-neutral world, all investors are happy discounting all cash flows at the risk-free rate.

If ΛV is a martingale it must be the case that

$$\lambda_0 V_0 = E(\Lambda_T V_T).$$

Thus,

$$V_0 = E\left(\frac{\Lambda_T \beta_T \beta_0}{\Lambda_0 \beta_0 \beta_T} V_T\right) = E\left(\frac{\mathcal{E}_T}{\mathcal{E}_0} e^{-\int_0^T r_s ds} V_T\right) = E^*\left(e^{-\int_0^T r_s ds} V_T\right). \quad (4)$$

Therefore, we can value any asset by discounting expected cash flows at the risk-free rate of return.

Example 1. The price of a zero-coupon bond paying 1 unit of consumption at time T is just

$$P(T) = E^*\left(e^{-\int_0^T r_s ds}\right),$$

since $V_T = 1$. Therefore, $e^{-\int_0^T r_s ds}$ acts like a discount factor under the risk-neutral measure. \square

Example 2. A futures contract is an obligation to purchase or sell an asset S for a pre-specified price namely the futures price at a specific date T in the future. The key feature of futures contracts is that the gains or losses are realized daily. Also, to buy or sell a futures there is no cash outflow. Even though in real markets investors need to deposit a small margin, for the purpose of pricing the futures we can assume that the margin amount is negligible.

Therefore, if we denote by dF the futures gains or losses in a long position from t to $t + dt$, it must be the case that

$$E(\Lambda dF) = 0,$$

since no cash is required to obtain a potential gain or loss of dF during the period. We can then re-write the previous expression for $0 \leq t \leq T$ as

$$0 = E\left(\frac{\Lambda_t \beta_t}{\Lambda_0 \beta_0} dF\right) = E^*(dF).$$

Thus, under the risk-neutral measure the futures price process must be a local martingale. For many models, we can actually write that the futures price is a martingale under the risk-neutral measure, implying

$$F(T) = E^*(S_T).$$

Even though sometimes it might be hard to show that the futures is a P^* -martingale, we can always compute $E^*(S_T)$ and verify that the futures satisfy the local martingale property. \square

Example 3. The forward price $\varphi(T)$ is the delivery price in a forward contract expiring at time T such that the value of the contract is zero, i.e.,

$$\mathbb{E}^* \left(e^{-\int_0^T r_s ds} (S_T - \varphi(T)) \right) = 0.$$

Therefore,

$$\varphi(T) = \frac{\mathbb{E}^* \left(e^{-\int_0^T r_s ds} S_T \right)}{\mathbb{E}^* \left(e^{-\int_0^T r_s ds} \right)} = \frac{\text{Cov}^* \left(e^{-\int_0^T r_s ds}, S_T \right)}{B(T)} + F(T).$$

The forward price is equal to the futures price plus the risk-neutral covariance between the risk-neutral discount factor and the underlying asset. Thus, the forward price is equal to the futures price only when this covariance is zero. \square

The Black-Scholes Model

The Black-Scholes formula to price options is one of the most important accomplishments in finance. A European call option gives its buyer the right but not the obligation to purchase an asset for a pre-determined price K at a future date T . Thus, the buyer of the call option pays K to receive a stock worth S_T only when $S_T > K$.

In their original model, Black and Scholes (1973) assumes that all parameters are constant. This implies that

$$\beta_T = \beta_0 e^{rT},$$

and

$$V_0 = e^{-rT} \mathbb{E}^*(V_T).$$

In the Black-Scholes model, the stock price process under the risk-neutral measure is

$$\frac{dS}{S} = r dt + \sigma dB^*.$$

We can solve for S_T to find

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma B_T^*}.$$

Thus, under the risk-neutral measure $\ln(S_T)$ is normally distributed with mean

$$E \ln(S_T) = \ln(S_0) + \left(r - \frac{1}{2}\sigma^2\right)T,$$

and variance

$$V \ln(S_T) = \sigma^2 T.$$

Example 4. The risk-neutral probability that the stock price S_T is greater than K at time T is

$$\begin{aligned} P^*(S_T > K) &= P^*(\ln(S_T) > \ln(K)) \\ &= P^*\left(Z > \frac{\ln(K) - \ln(S_0) - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma^2 T}\right) \\ &= P^*\left(Z < \frac{\ln(S/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma^2 T}\right), \end{aligned}$$

where Z denotes a standard normally distributed random variable. In the Black-Scholes model, we typically write

$$d_2 = \frac{\ln(S/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma^2 T},$$

and $N(d) = P^*(Z < d)$, so that $P^*(S_T > K) = N(d_2)$. □

For a given event $A \in \mathcal{F}$, the indicator function $\mathbf{1}_{\{A\}}(\omega)$ is equal to 1 if $\omega \in A$ and 0 otherwise. Thus, $\mathbf{1}_{\{S_T > K\}}$ is equal to 1 whenever $S_T > K$ and zero otherwise. The payoff of a call option can then be defined as

$$\text{Call Payoff} = (S_T - K)\mathbf{1}_{\{S_T > K\}} = S_T\mathbf{1}_{\{S_T > K\}} - K\mathbf{1}_{\{S_T > K\}}.$$

The price of a call must then be given by

$$C_0 = E\left(\frac{\Lambda_T}{\Lambda_0}(S_T - K)\mathbf{1}_{\{S_T > K\}}\right) = E\left(\frac{\Lambda_T}{\Lambda_0}S_T\mathbf{1}_{\{S_T > K\}}\right) - K E\left(\frac{\Lambda_T}{\Lambda_0}\mathbf{1}_{\{S_T > K\}}\right). \quad (5)$$

To compute the first expectation in (5), a nice trick is to realize that $\mathcal{E}^S = \Lambda S$ is a strictly positive

martingale defining a new measure P^S such that

$$\frac{dP^S}{dP} = \frac{\mathcal{E}_T^S}{\mathcal{E}_0^S}.$$

Thus,

$$E\left(\frac{\Lambda_T}{\Lambda_0} S_T \mathbf{1}_{\{S_T > K\}}\right) = S_0 E\left(\frac{\Lambda_T S_T}{\Lambda_0 S_0} \mathbf{1}_{\{S_T > K\}}\right) = S_0 E^S(\mathbf{1}_{\{S_T > K\}}) = S_0 P^S(S_T > K).$$

To compute the second expectation in (5) we can just use the risk-neutral measure

$$E\left(\frac{\Lambda_T}{\Lambda_0} \mathbf{1}_{\{S_T > K\}}\right) = \frac{\beta_0}{\beta_T} E\left(\frac{\Lambda_T \beta_T}{\Lambda_0 \beta_0} \mathbf{1}_{\{S_T > K\}}\right) = e^{-rT} E^*(\mathbf{1}_{\{S_T > K\}}) = e^{-rT} P^*(S_T > K).$$

The price of the call can then be written as

$$C_0 = S_0 P^S(S_T > K) - K e^{-rT} P^*(S_T > K).$$

To compute $P^S(S_T > K)$, we know that

$$B_t^S = B_t - \int_0^t \left(\frac{d\mathcal{E}^S}{\mathcal{E}^S}\right) (dB) = B_t + (\lambda - \sigma)t$$

is a Brownian motion under P^S . Thus,

$$\frac{dS}{S} = (r + \sigma^2) dt + \sigma dB^S.$$

We can follow the steps in Example 4 to conclude that

$$P^S(S_T > K) = N(d_1),$$

where

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma^2 T}.$$

To price a European put option we can proceed in a similar way. Remember that a European put

option gives its buyer the right but not the obligation to sell an asset for a pre-determined price K at a future date T . Therefore, the payoff of the European put at maturity is

$$\text{Put Payoff} = K \mathbf{1}_{\{S_T < k\}} - S_T \mathbf{1}_{\{S_T < k\}}.$$

The price P_0 of the put today is then given by

$$P_0 = K e^{-rT} P^*(S_T < K) - S P^S(S_T < K).$$

Thus, $P^*(S_T < K) = 1 - N(d_2) = N(-d_2)$ and $P^S(S_T < K) = 1 - N(d_1) = N(-d_1)$. We can summarize these results in the following property.

The Black-Scholes Model

In the Black-Scholes model, the prices C and P of European call and put options, respectively, are given by

$$C = SN(d_1) - Ke^{-rT}N(d_2),$$

$$P = Ke^{-rT}N(-d_2) - SN(-d_1),$$

where

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\ln(S/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$

and $N(d)$ denotes the cumulative probability that a standard normal random variable is less than d .

Partial Differential Equations in the Black-Scholes Model

The Black-Scholes formula was originally derived as the solution of a partial differential equation (PDE). It is indeed the case that any asset in the Black-Scholes model must satisfy the same PDE. Consider a derivative V that pays $f(S_T)$ at time T .

If V is a function of S and t , then Ito's lemma implies that

$$\begin{aligned}dV &= \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial V}{\partial t}dt \\ &= \left(rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt + \sigma S\frac{\partial V}{\partial S}dB^*.\end{aligned}$$

Equation (3) then implies that any derivative written on S must satisfy the following partial differential equation

$$rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} = rV.$$

It is in theory possible to solve the partial differential equation subject to a terminal value to price any derivative like a European call or put option written on the stock. In practice, it is easier to use a change of measure to find the value of the derivative.

References

Black, Fischer, and Myron Scholes. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (3): 637–54.