

The Treynor-Black Model

Model Setup

The model presented in this note is from Treynor and Black (1973). We have n risky assets with excess returns following a single-index model

$$R_i = \alpha_i + \beta_i R_M + e_i,$$

where $E(e_i) = 0$, $\text{Cov}(R_M, e_i) = 0$ and $\text{Cov}(e_i, e_j) = 0$ for $i \neq j$, for all $i, j = 1, \dots, n$. Suppose we form a portfolio P with the n risky assets and the market portfolio M .

Denote by w_i the weights of the n risky assets and w_M the weight in the market portfolio. Of course, the sum of the portfolio weights must be equal to one, i.e.,

$$\sum_{i=1}^n w_i + w_M = 1. \quad (1)$$

We can write the excess returns of this portfolio over the risk-free asset as

$$R_P = \alpha_P + \beta_P R_M + e_P.$$

The portfolio alpha includes the alpha of all the securities except the market that has no alpha. Therefore,

$$\alpha_P = \sum_{i=1}^n w_i \alpha_i. \quad (2)$$

We will see later that the alpha of the optimal portfolio is positive since its weights are positive for securities with positive alpha, and negative otherwise.

The beta of this portfolio is a weighted average of the beta of all securities and the beta of the market, which is one. Thus,

$$\beta_P = \sum_{i=1}^n w_i \beta_i + w_M. \quad (3)$$

For simplicity, in the following we will not replace immediately (3) into the expressions for the expected return and variance of the portfolio. It is apparent that choosing w_M given the weights of the other assets is the same as choosing β_P . Therefore, our optimization procedure will choose to maximize the Sharpe ratio of the target portfolio by choosing the weights in the individual assets and the beta of the target portfolio.

The idiosyncratic risk includes the firm-specific risks of all securities except the market that has no idiosyncratic risk. Because the firm-specific risks are uncorrelated with each other, we have that

$$\sigma^2(e_P) = \sum_{i=1}^n w_i^2 \sigma^2(e_i). \quad (4)$$

Thus,

$$E(R_P) = \sum_{i=1}^n w_i \alpha_i + \beta_P E(R_M), \quad (5)$$

and

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sum_{i=1}^n w_i^2 \sigma^2(e_i). \quad (6)$$

The expected return and variance in (5) and (6) depend on the alphas generated by the active securities and the systematic exposure of the resulting portfolio. This is the essence of the Treynor-Black model. The active part of the portfolio, which might behave like a zero-cost portfolio, generates an alpha but might increase the total risk by adding diversifiable risk. It might also increase the expected return by loading on systematic risk, and therefore increase the systematic risk of the resulting portfolio.

Solving the Model

The objective is to find the portfolio weights $w_1, w_2, \dots, w_n$ and w_M that maximize its Sharpe ratio. Since $\beta_P = \sum_{i=1}^n w_i \beta_i + w_M$, this is equivalent to finding $w_1, w_2, \dots, w_n$ and the beta of the final portfolio that maximize the Sharpe ratio of the target portfolio.

Maximizing the Sharpe ratio is equivalent to minimizing the variance of the portfolio for a given expected return. Of course, only one particular expected return satisfies the restriction established in (1). For other targets of the expected return, we would have to borrow or invest the difference at the risk-free rate. If we denote by μ_P a possible target expected return, the minimization problem is as follows:

$$\begin{aligned} \min_{\{w_1, w_2, \dots, w_n, \beta_P\}} \quad & \frac{1}{2} \sigma_P^2 \\ \text{s.t.} \quad & E(R_P) = \mu_P - r_f. \end{aligned}$$

The FOCs are

$$\begin{aligned} w_i \sigma^2(e_i) - \lambda \alpha_i &= 0, \forall i = 1, 2, \dots, n \\ \beta_P \sigma_M^2 - \lambda E(R_M) &= 0, \\ E(R_P) - (\mu_P - r_f) &= 0. \end{aligned}$$

The last FOC is not important since we will compute λ later by imposing the condition that the sum of the weights is equal to one. The first FOC implies that for each security we have that

$$w_i = \lambda \frac{\alpha_i}{\sigma^2(e_i)}. \quad (7)$$

The expression says that the weight of each security in the active portfolio increases with the alpha but decreases with the variance of the firm-specific risk. In other words, the weight of each security in the active portfolio depends on the *quality* of the alpha. A small alpha with little firm-specific variance can be a better opportunity than a large alpha with a large firm-specific variance. Assets with negative alphas carry a negative portfolio weight whereas assets with positive alphas have a positive weight in the final portfolio.

Using the weights of the individual assets, the weight in the market portfolio can be computed as

$$w_M = 1 - \lambda \sum_{i=1}^n \frac{\alpha_i}{\sigma^2(e_i)}. \quad (8)$$

The second FOC implies that

$$\beta_P = \lambda \frac{E(R_M)}{\sigma_M^2}. \quad (9)$$

On the other hand, from the definition of the portfolio beta we get that

$$\beta_P = \lambda \sum_{i=1}^n \frac{\alpha_i}{\sigma^2(e_i)} \beta_i + w_M.$$

Using (8) in the previous expression we get

$$\beta_P = \lambda \sum_{i=1}^n \frac{\alpha_i}{\sigma^2(e_i)} (\beta_i - 1) + 1. \quad (10)$$

Combining (9) with (10) we get

$$\lambda = \frac{1}{\frac{E(R_M)}{\sigma_M^2} + \sum_{i=1}^n \frac{\alpha_i}{\sigma^2(e_i)} (1 - \beta_i)}. \quad (11)$$

The previous equation provides an expression for λ that can be computed using the model inputs. The value for λ can then be used to compute the weights of the individual assets and of the market portfolio.

The Information Ratio

We can use (7) and (9) in (5) to get an expression of the expected return of the optimal portfolio:

$$E(R_P) = \lambda \left[\sum_{i=1}^n \left(\frac{\alpha_i}{\sigma(e_i)} \right)^2 + \left(\frac{E(R_M)}{\sigma_M} \right)^2 \right]. \quad (12)$$

Similarly, (6) allows us to compute the optimal portfolio variance, i.e.,

$$\sigma_P^2 = \lambda^2 \left[\sum_{i=1}^n \left(\frac{\alpha_i}{\sigma(e_i)} \right)^2 + \left(\frac{E(R_M)}{\sigma_M} \right)^2 \right]. \quad (13)$$

These two expressions above yield the following relationship between the Sharpe ratio of the optimal portfolio to the market Sharpe ratio,

$$\left(\frac{E(R_P)}{\sigma_P} \right)^2 = \sum_{i=1}^n \left(\frac{\alpha_i}{\sigma(e_i)} \right)^2 + \left(\frac{E(R_M)}{\sigma_M} \right)^2. \quad (14)$$

The equation shows that the Sharpe ratio of the optimal portfolio is higher than the Sharpe ratio of the market alone. Each security contributes positively in increasing the Sharpe ratio of the optimal portfolio, regardless of whether the alpha is positive or negative. Remember that securities with negative alphas have negative weights, i.e., they are sold short in order to take advantage of the fact that they are expensive relative to the model.

For a given security i , the term $\alpha_i/\sigma(e_i)$ is its **information ratio**, introduced in [The Index Model](#), and it measures how *good* the alpha is. If the volatility of the firm-specific risk is small, then it will be easier to diversify that risk away and hence extract the alpha of the security. Each security contributes to increasing the squared value of the Sharpe ratio of the optimal portfolio by the square of its information ratio.

Practice Problems

These problems give you a chance to practice the concepts introduced in this chapter. Try to solve each one on your own before expanding the solution.

Solutions to all problems can be found at <https://lorenzozaranjo.com/class-materials/investments/>.

Unless stated otherwise, the market has an expected excess return of $E(R_M) = 8\%$ and a volatility of $\sigma_M = 20\%$, so its Sharpe ratio is 0.40.

Problem 1 (Adding One Stock to the Market). Olivia Carter, a portfolio manager at Lakemont Investors, holds the market and believes Verdant Foods (VDF) has an alpha of 3%, a beta of 1.2, and a residual volatility of 30%.

- Use (11) to compute λ , and find the weights in VDF and in the market.
- Compute the beta, expected excess return, and volatility of the optimal portfolio, and its Sharpe ratio.
- Check the Sharpe ratio with (14).

Problem 2 (Profiting from a Negative Alpha). Olivia's analysts now conclude that Brackley Retail (BRK) is overpriced: it has an alpha of -2% , a beta of 0.9, and a residual volatility of 25%. Consider a portfolio of BRK and the market only.

- Find λ , the weight in BRK, and the weight in the market.
- Does BRK raise or lower the Sharpe ratio of the portfolio? By how much?

Problem 3 (A Two-Stock Active Portfolio). Nathan Price at Westbury Capital combines the market with two stocks: Arlo Systems (ARS), with an alpha of 2%, a beta of 1.2, and a residual volatility of 25%; and Belmont Paper (BMP), with an alpha of -1% , a beta of 0.8, and a residual volatility of 20%.

- Compute λ and the weights in ARS, BMP, and the market.
- Compute the beta of the optimal portfolio in two ways, using (9) and using the weights.
- Compute the Sharpe ratio of the optimal portfolio.

Problem 4 (Same Alpha, Different Risk). Two analysts at Westbury Capital each pitch a stock with an alpha of 2%. Cresswell Labs (CRL) has a residual volatility of 10%, and Duncan Exploration (DXP) has a residual volatility of 40%.

- How do the optimal weights of the two stocks compare?
- By how much does each stock raise the squared Sharpe ratio of the optimal portfolio?

Problem 5 (How Many Ideas Does a Fund Need?). Harriet Lowe is launching a fund that holds the market plus an active portfolio. Her research team can find stocks with information ratios of 0.05 in absolute value, independent of each other. How many such stocks does she need for the fund's Sharpe ratio to reach 0.50?

Problem 6 (The Cost of False Alphas). Return to Nathan's portfolio in [Exercise 3](#). Suppose his analysts were wrong, and the true alphas of ARS and BMP are both zero.

- a. Using the same weights, what are the portfolio's expected excess return and volatility?
- b. Compare its Sharpe ratio with that of the market. What does the comparison say about acting on alpha forecasts?

Problem 7 (Monthly and Annual Information Ratios). A stock has an alpha of 0.25% per month and a residual volatility of 5% per month, with residuals independent from month to month. Compute its monthly and its annual information ratio.

Problem 8 (Choosing Between Two Managers). The Corliss Foundation holds the market and wants to add one market-neutral manager. Manager Xu promises an alpha of 4% with a residual volatility of 20%. Manager Yates promises an alpha of 2% with a residual volatility of 5%.

- a. Compute each manager's information ratio, and the Sharpe ratio the foundation could reach by combining the market with each manager optimally.
- b. A trustee prefers Xu because his alpha is twice as large. How would you respond?

Problem 9 (What Does a Sharpe Ratio Reveal?). A fund that holds the market plus an optimal active portfolio reports a Sharpe ratio of 0.55. What is the information ratio of its active portfolio, assuming the market's Sharpe ratio is 0.40?

References

Treynor, Jack L., and Fischer Black. 1973. "How to Use Security Analysis to Improve Portfolio Selection." *Journal of Business* 46 (1): 66–86.