The Fisher Model

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Introduction

The theory of finance is concerned with how investors allocate resources over time.[1] Investors must decide today how much to save, how much to consume, and how to invest their savings. Therefore, investment theory aims to determine the best way to maximize the benefit that investors derive from their consumption today and tomorrow. We denote this consumption bundle by $\{C_0, C_1\}$.

Utility Theory

We start this notebook by thinking about how investors can rank consumption bundles. In economics, a very convenient way to rank consumption bundles is to use a *utility function*. The idea of a utility function is to assign a real number to each consumption bundle. A consumption bundle is then preferred to another if the utility number is larger.

For example, consider the following function,

$$U(C_0, C_1) = \ln(C_0) + \ln(C_1).$$

We can compute $U(3, 2) = 1.79$ and $U(2.5, 2.5) = 1.83$ which shows that this agent prefers consuming 2.5 units today and tomorrow over consuming three units today and two units tomorrow. For this consumer, we have that $(2.5, 2.5) \gtrsim (3, 2)$. The value of the utility function

[1] For those interested, a more detailed explanation of the topics covered in this notebook can be found in Fama and Miller (1972).

is irrelevant since applying any increasing function to a utility function will not change the rankings of consumption bundles.[2]

From the previous example, we can see that the consumer will be indifferent to some consumption bundles. For example, $U(2,1) = U(1,2) = 0.69$, which we denote by $(1,2) \sim (2,1)$. The set of all consumption bundles that provide the same utility is called an **indifference curve**.

The figure below shows three different indifference curves. The blue line denotes all consumption bundles with a utility equal to U_1. The orange curve denotes all bundles that provide a utility equal to $U_2 > U_1$. The green line provides an even higher level of utility $U_3 > U_2$.

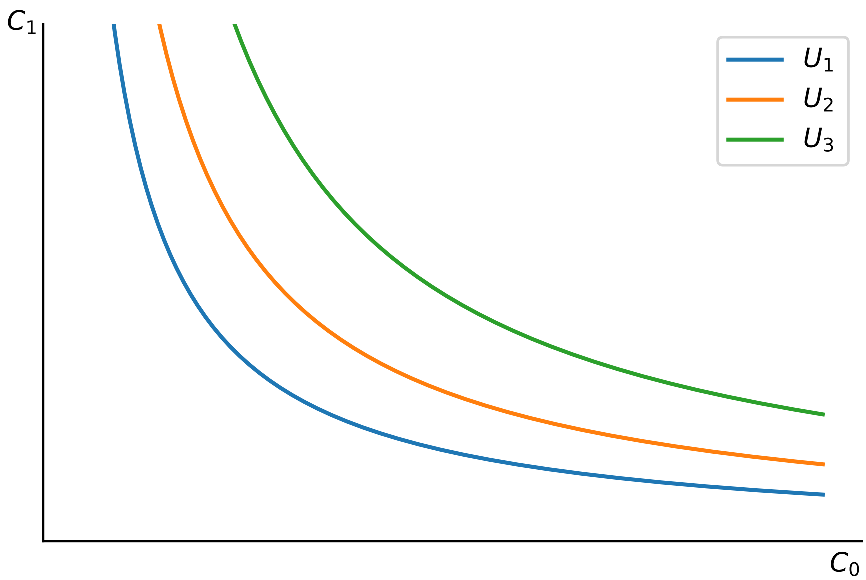

Figure 1: The figure shows three indifference curves.

The figure also displays something that we expect to find in real life: utility should increase with consumption. In the previous example, we found that $(1,2) \sim (2,1)$, but of course we would expect $(1,2) \precsim (1,3)$. In other words, the *marginal utility* of consumption must be positive for consumption in both periods, i.e., $\frac{\partial U}{\partial C_i} > 0$ for $i = \{0,1\}$.

[2]Since $U(C_0, C_1) = \ln(C_0 C_1)$, the new utility function $V(C_0, C_1) = C_0 C_1$ generates the same rankings of consumption bundles.

Marginal utility should also decrease with consumption, i.e., $\frac{\partial^2 U}{\partial C_i^2} < 0$ for $i = \{0, 1\}$, since each additional unit of consumption can only increase utility at a lower rate. The first unit of consumption provides a much larger increase in utility than the last.

Example 1. In finance, it is common to use separable utility functions of the form

$$U(C_0, C_1) = u(C_0) + \beta u(C_1).$$

The choice

$$u(C) = \begin{cases} \frac{C^{1-\gamma}-1}{1-\gamma}, & \text{if } \gamma \geq 0, \gamma \neq 1 \\ \ln(C), & \text{if } \gamma = 1 \end{cases}$$

is called **power utility** if $\gamma \neq 1$ and log utility if $\gamma = 1$. Another common choice for $u(C)$ is

$$u(C) = -e^{-aC},$$

usually called **exponential utility**.

We can compute the utility differential as

$$dU = \frac{\partial U}{\partial C_0} dC_0 + \frac{\partial U}{\partial C_1} dC_1.$$

Since an indifference curve keeps the utility level constant, for all points in the indifference curve, we have that $dU = 0$, implying that

$$\frac{dC_1}{dC_0} = -\frac{\frac{\partial U}{\partial C_0}}{\frac{\partial U}{\partial C_1}}. \tag{1}$$

The absolute value of the derivative of C_1 with respect to C_0 is called the **marginal rate of substitution** (MRS) between C_1 and C_0. The MRS compares how important it is to consume tomorrow versus today at any given point.

Production Functions

An investor must first decide how much to consume today and how much to save for the next period. Two factors determine this decision. On the one hand, the MRS determines how future consumption feels compared to current consumption. On the other hand, the ability to transform current consumption into future consumption is essential in deciding how much to consume today versus tomorrow.

We model the ability to convert current consumption into future consumption through a *production function*. All consumers start with a certain level of wealth, W, measured in terms of current consumption. Consumers can then decide how much to consume today, given by C_0^*, and how much to invest. An investment of $K = W - C_0^*$ will generate $C_1^* = f(K)$ of consumption tomorrow.

The production function combines all available investment projects and ranks them from best to worse in terms of return. Of course, if you have little to invest you want to use it in projects that have the best profitability. For example, consider the following portfolio of investment opportunities, ranked by **internal rate of return** (IRR).

Project	Maximum Investment	IRR
I	1	500%
II	3	300%
III	5	100%
IV	11	0%

Project I has a maximum investment of one unit of consumption and generates five additional units per unit invested. Thus, Project I transforms one unit of consumption today into six units of consumption next period. Project II transforms each additional unit of consumption today into four units of consumption tomorrow. Thus, investing four units of consumption in projects I and II generates $f(4) = 1 \times 6 + 3 \times 4 = 18$ units of consumption tomorrow.

Assuming that we can invest fractions of today's consumption, we can then generate the following production function $f(K)$.

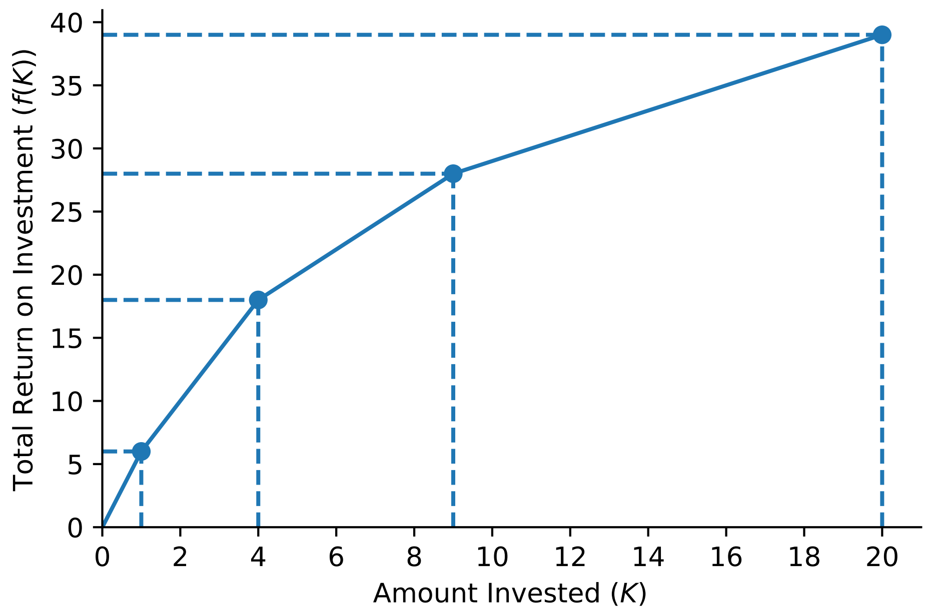

Figure 2: The figure shows a piece-wise linear production function.

The production function we just built is continuous in its range of definition $K \in [0, 20]$. It is also increasing in K as long as we assume limited liability, which would be the case if you incorporate your productive activities as a firm. Note that Project IV has a net return of 0%, which means that each unit of consumption invested generates one unit of consumption tomorrow. The worst possible scenario under limited liability is that the IRR of the project is -100%. In that case all additional units invested in such a project would be destroyed, at which point the production function would be flat.

The production function in Figure 2 is also concave, which is a consequence of investing in the projects with better profitability first. Project I is the best in terms of profitability. If we only have one unit of consumption to invest we should clearly choose it. Project III will be chosen only after four units have been invested in projects I and II.

Typically, we assume that the production function is smooth such that $f'(K) > 0$ and $f''(K) < 0$, which yields a continuous, increasing, and concave function. A production function with such properties is consistent with our previous analysis.

In the following, it is useful to express the function in terms of $K = W - C_0$, so that $C_1 =$

$f(W - C_0)$. If the consumer decides to invest nothing and consume everything today, we have that $K = 0$ and $C_0^* = W$. If, on the other hand, the consumer decides to consume nothing today and invest everything, then we have that $K = W$ and $C_0^* = 0$.

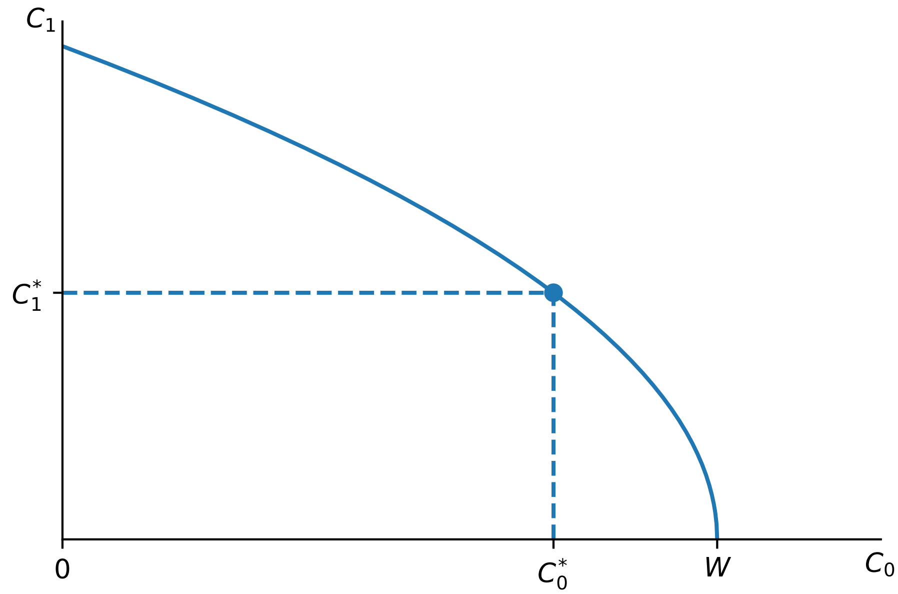

Figure 3: The figure shows the investment opportunity set available to an investor.

Maximizing Utility

Consider now an investor with utility function $U(C_0, C_1)$ and initial wealth W. The investor has the ability to invest $K = W - C_0$ into a production function that yields next period $C_1 = f(K)$. We can write the investor's problem as follows

$$\max_{\{C_0, C_1\}} U(C_0, C_1)$$
$$\text{s.t. } C_1 = f(W - C_0)$$

To solve the previous optimization problem, we can write the Lagrangian as

$$\mathcal{L} = U(C_0, C_1) - \lambda(C_1 - f(W - C_0)).$$

The first-order conditions (FOC) are

$$\frac{\partial \mathcal{L}}{\partial C_0} = \frac{\partial U}{\partial C_0} - \lambda f'(W - C_0) = 0,$$
$$\frac{\partial \mathcal{L}}{\partial C_1} = \frac{\partial U}{\partial C_1} - \lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = C_1 - f(W - C_0) = 0.$$

The first two FOCs imply that the marginal rate of substitution (MRS) must be equal to the marginal rate of transformation (MRT) between C_1 and C_0. The last FOC says that whatever the investor does not consume today is invested and can be consumed tomorrow to yield $C_1 = f(W - C_0)$.

The figure below shows the optimal consumption choice. The indifference curve at the optimum is tangent to the production function, meaning that the MRS of the consumer equalizes the MRT provided by the technology.[3]

Example 2. Consider an investor with utility $U(C_0, C_1) = \ln(C_0) + \ln(C_1)$. The investor has initial wealth $W = 1$ and can invest $K = 1 - C_0$ in a technology that produces $f(K) = \sqrt{K}$ next period.

The investor maximizes her utility if her consumption (C_0^*, C_1^*) satisfies

$$\text{MRS} = \frac{C_1}{C_0} = \frac{1}{2\sqrt{1 - C_0}} = \text{MRT}.$$

Since $C_1 = \sqrt{1 - C_0}$, we have that

$$\frac{\sqrt{1 - C_0}}{C_0} = \frac{1}{2\sqrt{1 - C_0}},$$

[3] To guarantee a unique optimum, the indifference curve must be convex. Quasi-concave utility functions generate convex upper-contour sets defined as $\{x \in X : U(x) \geq c\}$, where $X \in \mathbb{R}^{\mathbb{N}}$ is the consumption set. In our case $X = \mathbb{R}^{2+}$. It can be shown that a function is quasi-concave if and only if $U(\lambda x + (1-\lambda)y) \geq \min(U(x), U(y))$, where $x, y \in X$ and $0 \leq \lambda \leq 1$.

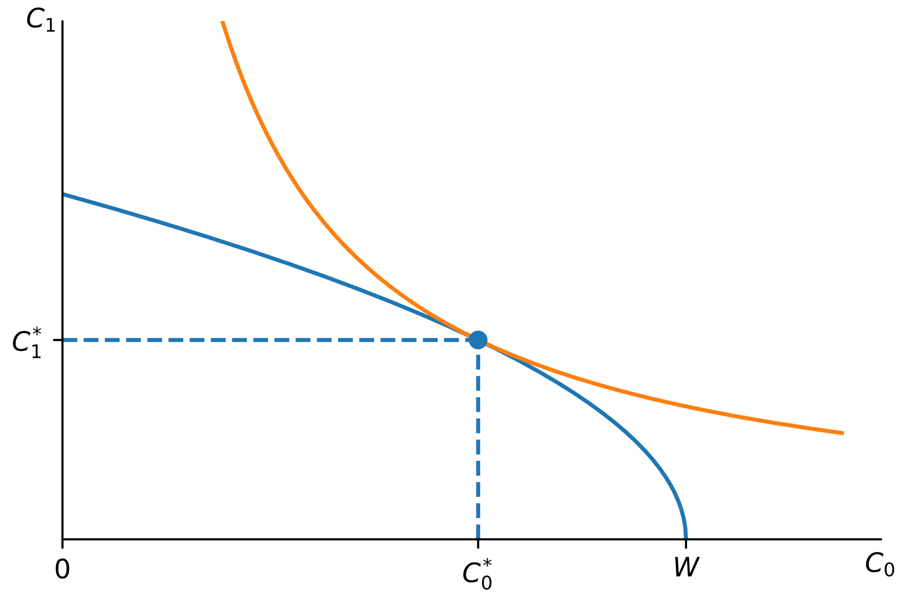

Figure 4: The figure shows the optimal consumption choice given a production function and initial wealth W.

which implies that $C_0 = \frac{2}{3}$ and $C_1 = \sqrt{\frac{1}{3}}$.

In the previous example, we have that $C_0 + C_1 > W = 1$, meaning that the production function improves the utility of the consumer compared to a simple storing technology that only allows to save consumption for later.

The Role of Capital Markets

The Production Decision

Investors can do better than autarky if they organize their economy differently. Let's delegate the production decision to a manager with access to a technology $f(K)$. Furthermore, assume that consumers have access to capital markets where they can borrow or lend at an interest rate r. The manager is given a certain amount of wealth W, and must decide how much to sell today, investing the rest in the technology for future production, which we denote by (Q_0, Q_1), respectively.

Shareholders expect the manager to choose (Q_0, Q_1) to maximize the value of the firm

$$V = Q_0 + \frac{Q_1}{1+r}.$$

The manager faces the budget constraint that he can only invest what the firm does not sell today, i.e., $K = W - Q_0$. The problem that the manager must solve is given by

$$\max_{\{Q_0\}} Q_0 + \frac{f(W - Q_0)}{1+r}.$$

The FOC is

$$\text{MRT} = f'(W - Q_0^*) = 1 + r.$$

The figure below shows that the optimal production choice (Q_0^*, Q_1^*) is such that at that point, the production function is tangent to the **capital market line** (CML) whose slope coefficient is $-(1+r)$.

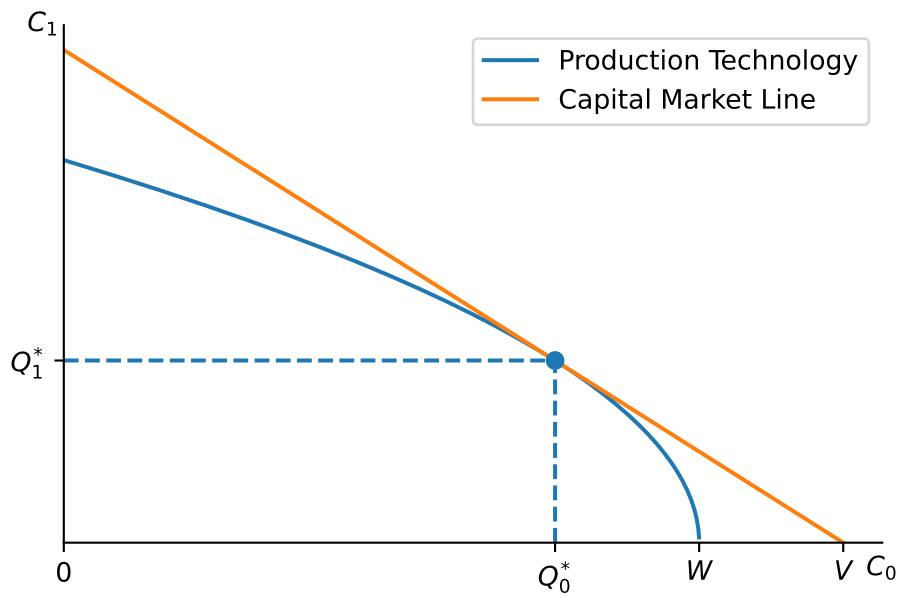

Figure 5: The figure shows the optimal production policy for the firm given a production function and initial wealth W.

The intercept of the CML with the x-axis determines the firm value. By choosing the tangency point between the two lines, the manager maximizes the firm's value by selecting the intercept that is furthest to the right. To increase the firm's size, the manager would need a more significant initial investment of W.

The difference between V and W is the **net present value** (NPV) created using the technology. By investing an initial capital of W, shareholders now have an asset worth more than the initial investment. The manager should then accept all projects with positive NPVs.

Example 3. Consider the same production function of Example 2, i.e., $f(K) = \sqrt{K}$ and again take $W = 1$. The market interest rate is r. The policy (Q_0^*, Q_1^*) that maximizes firm-value is such that

$$\frac{1}{2\sqrt{1 - Q_0^*}} = 1 + r,$$
$$Q_1^* = \sqrt{1 - Q_0^*}.$$

Thus, $Q_1^* = \frac{1}{2(1+r)}$ and $Q_0^* = 1 - \frac{1}{4(1+r)^2}$. The value of the firm is then

$$V = 1 - \frac{1}{4(1+r)^2} + \frac{1}{2(1+r)^2} = 1 + \frac{1}{4(1+r)^2},$$

which shows that the NPV of the technology is $\frac{1}{4(1+r)^2} > 0$.

Without access to capital markets, the value of the production technology is just W since this amount today can generate all possible production bundles (Q_0, Q_1). The CML is then another production function that gives investors access to superior bundles. According to the CML, the value of the technology is $V > W$.

The Consumption Decision

In the model, shareholders agree on how the firm should maximize its value, regardless of their utility for today's and future consumption. An investor with initial wealth W can create the previous firm, hire a manager, and incorporate the firm. The firm will then produce Q_0^* today, invest $K = W - Q_0^*$ and produce $f(K) = Q_1^*$ for consumption next period.

The investor could sell the firm for V, which can be used to consume C_0 today and invest the rest to consume $C_1 = (V - C_0)(1 + r)$ next period. The investor's problem is

$$\max_{\{C_0, C_1\}} U(C_0, C_1),$$
$$\text{s.t. } C_1 = (V - C_0)(1 + r).$$

The Lagrangian of this problem is

$$\mathcal{L} = U(C_0, C_1) - \lambda(C_1 - (V - C_0)(1 + r)),$$

implying the following FOC conditions

$$\frac{\partial \mathcal{L}}{\partial C_0} = \frac{\partial U}{\partial C_0} - \lambda(1 + r) = 0,$$
$$\frac{\partial \mathcal{L}}{\partial C_1} = \frac{\partial U}{\partial C_1} - \lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = C_1 - (V - C_0)(1 + r) = 0.$$

The first two FOCs imply that the MRS for the consumer is equal to the rate of return of the CML,i.e.,

$$\text{MRS} = \frac{\frac{\partial U}{\partial C_0}}{\frac{\partial U}{\partial C_1}} = 1 + r.$$

The last FOC says that the present value of today's and future consumption must equal V, i.e.,

$$V = C_0 + \frac{C_1}{1 + r}.$$

Therefore, we have separated the decision of producing (Q_0^*, Q_1^*) given an initial wealth W, from the decision of consuming (C_0^*, C_1^*) given that the optimal production decision generates a present value of V that can be used to consume today and next period.

The figure below shows that the optimal consumption bundles that can be achieved by two investors with different marginal rates of substitution of consumption. Investor A prefers to give up consumption today in order to consume more next period. With access to capital markets, she now has an initial wealth of V, that allows her to save at a better marginal rate of return than with the production function alone. Investor B, on the other hand, prefers to borrow and consume more today by giving up consumption tomorrow. The existence of good functioning capital markets allows her to borrow at a cheaper rate of interest than the one provided by the production technology.

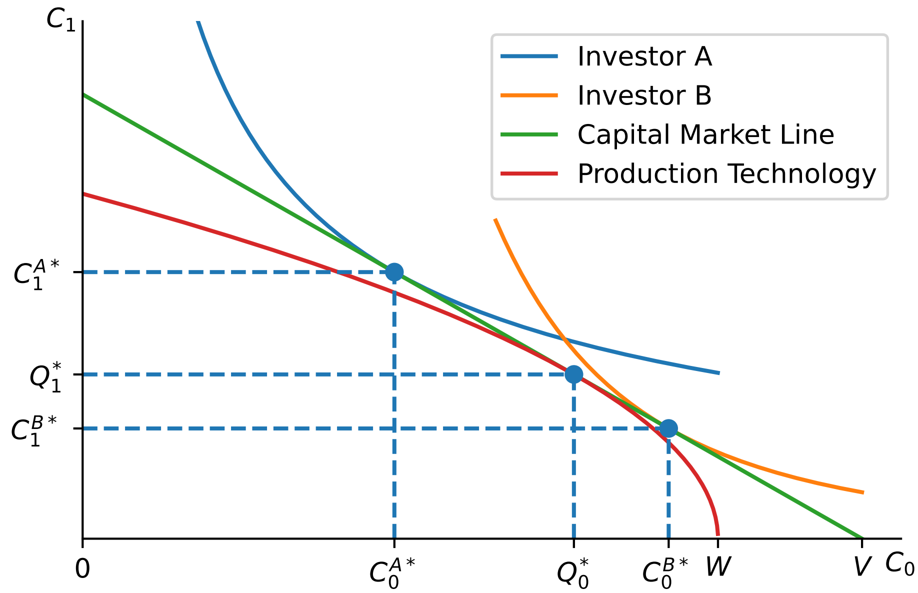

Figure 6: The figure shows the optimal production policy for the firm and optimal consumption decisions for two investors given a production function and initial wealth W.

Example 4. Consider an investor with initial wealth W who owns the technology function of Example 3. By producing $Q_0^* = 1 - \frac{1}{4(1+r)^2}$ and $Q_1^* = \frac{1}{2(1+r)}$, she maximizes the firm value at $V = 1 + \frac{1}{4(1+r)^2}$.

Assume that the investor has a utility function of the form $U(C_0, C_1) = \ln(C_0) + \beta \ln(C_1)$. If $\beta > 1$, the investor values more consumption tomorrow than today, whereas a $\beta < 1$ implies that the investor discounts future consumption relative to today's consumption.

The investor must decide how much to consume today and next period given the value of her equity. At the optimum, the investor equalizes her MRS with the total return on investment so that

$$\frac{C_1}{\beta C_0} = 1 + r.$$

Since $C_1 = (1+r)(V - C_0)$, we obtain $C_0^* = \frac{1}{1+\beta}V$ and $C_1^* = (1+r)\frac{\beta}{1+\beta}V$.

Fisher Separation Theorem

The previous analysis suggests that we can separate the firm's investment decision, which involves deciding how much to sell today and how much to reinvest to sell tomorrow, from the investment decision faced by the consumer. This separation result is known as the **Fisher Separation Theorem** after economist Irvin Fisher. Well-functioning capital markets play a crucial aspect in creating this separation.

All consumers are better off when firms maximize their values by undertaking positive NPV projects. It is the role of the firm's manager to ensure that companies maximize their values to shareholders. Corporate finance typically studies corporate policy and firm valuation.

Given shares of these profit-maximizing firms, consumers can choose how much to invest in each firm. Investment theory explores how economic agents can optimally decide how to allocate their resources, which is what we will study in this class. We will take the investment opportunity set as given and analyze how investors can maximize their utilities.

The CML determines an equilibrium for both firms and investors. In asset pricing we typically analyze the equilibrium pricing of securities from the investors' point of view, and it is usually called **consumption-based asset pricing**. The analysis in the previous section suggests that we could also study the equilibrium from the firms' point of view, which is usually called **production-based asset pricing**.

Different Borrowing and Lending Rates

The Fisher separation theorem relies on the fact that capital markets allow investors to lend and borrow at the same interest rate. This assumption allows us to write that the value of consuming C_0 today and C_1 tomorrow is the present value of the cash flows, i.e.,

$$V = C_0 + \frac{C_1}{1+r}.$$

Typically, the interest rate at which investors can borrow is higher than the rate at which they can lend. In this case, the CML extends to the left with slope $-(1+r_L)$ and to the right with slope $-(1+r_B)$, where r_L and r_B denote the lending and borrowing rates, respectively.

The figure below shows the resulting investment opportunity set. At the point (Q_0^{L*}, Q_1^{L*}) the MRT is equal to $1+r_L$, whereas at the point (Q_0^{B*}, Q_1^{B*}) it is $1+r_B$.

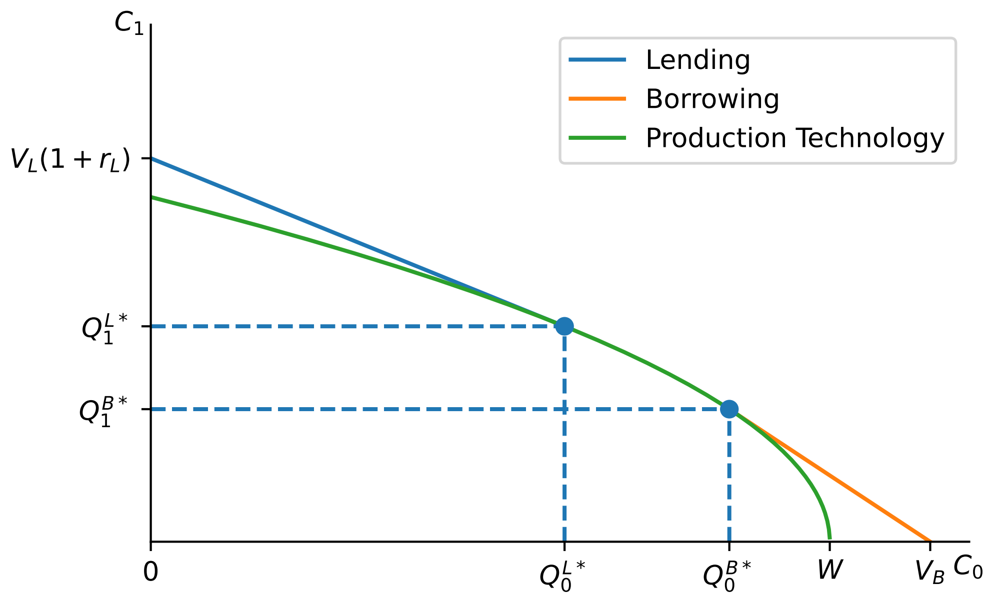

Figure 7: The figure shows the optimal production policy for a firm when investors face different lending and borrowing rates.

There are three possibilities for the investment opportunity set:

i. An investor willing to consume $C_0 < Q_0^{L*}$ can invest $V_L - C_0$ at r_L to consume $C_1 = (V_L - C_0)(1+r_L)$ next period, where $V_L = Q_0^{L*} + \frac{Q_1^{L*}}{1+r_L}$. Note that the y-intercept of this CML is $V_L(1+r_L)$.

ii. An investor choosing to consume C_0 such that $Q_0^{L*} \leq C_0 \leq Q_0^{B*}$ can consume $C_1 = f(W - C_0)$ next period, where $f(\cdot)$ denotes the production function.

iii. An investor willing to consume $C_0 > Q_0^{B*}$ can borrow $C_0 - Q_0^{B*}$ and consume $C_1 = (V_B - C_0)(1+r_B)$ next period, where V_B is defined analagously.

Having different borrowing and lending rates destroys the linearity of discounting and compounding. The present value of the production technology is V_B whereas the future value of it is

$V_L(1 + r_L)$.

The previous analysis shows how frictions can make asset pricing problems harder to solve. In the following, we will usually assume that markets are perfect and that borrowing and lending rates are the same.

References

Fama, E., and M. H. Miller. 1972. *The Theory of Finance*. Holt Rinehart & Winston.