Affine Recursive Utility in Continuous Time

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Introduction

The Recursive Utility in Continuous Time notebook developed the general continuous-time Epstein-Zin framework. This notebook specializes that framework to a simple one-factor Gaussian state model. The motivation is the same as in the discrete-time long-run risk literature: once expected growth depends on a persistent state variable, continuation utility becomes state dependent, and the pricing kernel reflects news about future investment opportunities as well as current consumption.

The goal is to have an example that is simple enough to show how the model works without losing the distinctive recursive-utility channel. The natural guess is that continuation utility is exponential-affine in the state, just as log valuation ratios are approximated by affine functions in discrete time. In continuous time this guess is not an exact solution to the primitive Epstein-Zin HJB, but it does deliver a useful local approximation. The payoff is a tractable system for the continuation-value coefficients and an explicit approximate SDF, linking the continuous-time recursive-utility setup of Duffie and Epstein (1992) to the affine approximation logic used in Campbell and Viceira (1999) and Bansal and Yaron (2004).

As in the additive-utility sequel, the primitives here are the consumption and state dynamics. Wealth is not specified independently. Instead, aggregate wealth is the value of the claim to the consumption stream, so we solve first for the wealth-consumption ratio and then derive the implied equilibrium wealth return process.

A One-Factor Affine-State Model

Consider the one-factor Gaussian system

$$d \ln C_t = (\mu_0 + \mu_1 x_t)\, dt + \sigma_c\, dB_t,$$

$$dx_t = \xi(\bar{x} - x_t)\, dt + \sigma_x\, dB_t,$$

where x_t is an Ornstein-Uhlenbeck state variable. Conditional on x_t, log consumption growth is Gaussian, but its conditional mean varies over time through a persistent state.

Continuation Value

This is the continuous-time counterpart of the discrete-time long-run risk setup. Conjecture a homothetic value function

$$V(W, x) = \frac{W^{1-\gamma}}{1-\gamma} h(x),$$

where $h(x)$ is the state-dependent continuation-value component. For the exact nonlinear problem, h is left unrestricted and will be solved from the HJB. The exponential-affine specification $h(x) = e^{a+bx}$ is introduced later as a local approximation.

Homothetic models are naturally expressed in ratios. Let

$$p(x_t) \equiv \frac{W_t}{C_t}$$

denote the wealth-consumption ratio. In a homothetic problem it depends only on the state, not on the level of wealth itself. The first-order condition from the theory notebook implies

$$f_c(C_t, V_t) = V_W(W_t, x_t),$$

which for Epstein-Zin preferences and the homothetic value function implies [1]

$$p(x) = \delta^{-\psi} h(x)^{\psi/\theta}.$$

Thus the continuation value determines the wealth-consumption ratio directly.

In equilibrium, aggregate wealth is the value of the claim to the consumption stream, so

$$W_t = C_t p(x_t).$$

The exact equilibrium return on wealth is therefore implied by the consumption process and the function $p(x)$; it is not specified independently.

Property 1 (Recursive Utility Wealth-Consumption Ratio)**.** *Under homothetic Epstein-Zin preferences,*

$$p(x) \equiv \frac{W_t}{C_t} = \delta^{-\psi} h(x)^{\psi/\theta}.$$

The state dependence of continuation value therefore translates directly into state dependence of the equilibrium wealth-consumption ratio.

Exact Nonlinear Solution

If one wants an exact solution, the natural route is numerical. Dropping the affine restriction on h, define

$$q(x) \equiv \ln h(x), \qquad \alpha \equiv \frac{\psi}{\theta}, \qquad \theta \equiv \frac{1-\gamma}{1-1/\psi}.$$

Since equilibrium wealth satisfies

$$W_t = C_t p(x_t) = \delta^{-\psi} C_t e^{\alpha q(x_t)},$$

[1] For Epstein-Zin preferences, $f_c(C_t, V_t) = \delta C_t^{-1/\psi}[(1-\gamma)V_t]^{1-1/\theta}$. With $V(W, x) = h(x)W^{1-\gamma}/(1-\gamma)$, we have $(1-\gamma)V_t = h(x_t)W_t^{1-\gamma}$ and $V_W(W_t, x_t) = h(x_t)W_t^{-\gamma}$. Substituting into the first-order condition gives $\delta C_t^{-1/\psi}[h(x_t)W_t^{1-\gamma}]^{1-1/\theta} = h(x_t)W_t^{-\gamma}$. The powers of W_t then cancel by homotheticity, leaving an equation in the wealth-consumption ratio alone, which yields $W_t/C_t = \delta^{-\psi}h(x_t)^{\psi/\theta}$.

Ito's lemma implies

$$d \ln W_t = \left[\mu_0 + \mu_1 x_t + \alpha q'(x_t)\xi(\bar{x} - x_t) + \frac{1}{2}\alpha q''(x_t)\sigma_x^2\right] dt + [\sigma_c + \alpha q'(x_t)\sigma_x]\, dB_t.$$

Hence the exact equilibrium wealth dynamics are

$$\frac{dW_t}{W_t} = \mu_W(x_t)\, dt + \sigma_W(x_t)\, dB_t,$$

where

$$\sigma_W(x) = \sigma_c + \alpha q'(x)\sigma_x,$$

and

$$\mu_W(x) = \mu_0 + \mu_1 x + \alpha q'(x)\xi(\bar{x} - x) + \frac{1}{2}\alpha q''(x)\sigma_x^2 + \frac{1}{2}\sigma_W(x)^2.$$

Substituting this equilibrium wealth process into the HJB and dividing through by V yields the exact nonlinear ODE

$$0 = \theta\left(\delta^{\psi} e^{-\alpha q(x)} - \delta\right) + (1-\gamma)(\mu_0 + \mu_1 x) + \psi\xi(\bar{x} - x)q'(x)$$

$$+\frac{1}{2}\psi\sigma_x^2 q''(x) + \psi(1-\gamma)\sigma_c\sigma_x q'(x) + \frac{1}{2}\psi^2\sigma_x^2 q'(x)^2 + \frac{1}{2}(1-\gamma)^2\sigma_c^2. \qquad (1)$$

This is a one-dimensional nonlinear boundary-value problem for $q(x)$, so the model remains numerically tractable even though it is not analytically affine.

Property 2 (Exact Wealth Dynamics). *If $q(x) = \ln h(x)$ solves the nonlinear ODE (1), then equilibrium wealth satisfies*

$$W_t = C_t p(x_t) = \delta^{-\psi} C_t e^{\alpha q(x_t)},$$

with dynamics

$$\frac{dW_t}{W_t} = \mu_W(x_t)\, dt + \sigma_W(x_t)\, dB_t,$$

where

$$\sigma_W(x) = \sigma_c + \alpha q'(x)\sigma_x,$$

and

$$\mu_W(x) = \mu_0 + \mu_1 x + \alpha q'(x)\xi(\bar{x} - x) + \frac{1}{2}\alpha q''(x)\sigma_x^2 + \frac{1}{2}\sigma_W(x)^2.$$

As with any second-order ODE, solving for q requires two boundary conditions. Rather than conditions at finite endpoints, the natural requirement here is asymptotic regularity: explosive solutions are ruled out by demanding that the slope of q vanish in the tails of the state space,

$$\lim_{x\to-\infty} q'(x) = 0, \qquad \lim_{x\to+\infty} q'(x) = 0.$$

These conditions eliminate the growing modes of the linearized ODE and select the unique stationary solution. They are implemented numerically in the companion notebook Exact Value Function for Recursive Utility.

Local Affine Approximation

To obtain closed-form intuition, return now to the exponential-affine guess

$$h(x) = e^{a+bx}.$$

Then

$$p(x) = \delta^{-\psi} e^{(\psi/\theta)(a+bx)},$$

and the exact equilibrium wealth volatility becomes constant:

$$\sigma_w \equiv \sigma_c + \frac{\psi}{\theta} b\, \sigma_x.$$

Substituting

$$V(W, x) = \frac{W^{1-\gamma}}{1-\gamma} e^{a+bx}$$

into the HJB gives [2]

$$0 = f\big(W/p(x), V(W,x)\big) + V_W\, W \left[\mu_0 + \mu_1 x + \frac{\psi}{\theta} b\, \xi(\bar{x} - x) + \frac{1}{2}\sigma_w^2\right]$$

$$+\frac{1}{2} V_{WW} W^2 \sigma_w^2 + V_x \xi(\bar{x} - x) + \frac{1}{2} V_{xx} \sigma_x^2 + V_{Wx} W \sigma_w \sigma_x.$$

Because

$$\frac{V_x}{V} = b, \qquad \frac{V_{xx}}{V} = b^2, \qquad \frac{V_W\, W}{V} = 1{-}\gamma, \qquad \frac{V_{WW} W^2}{V} = -\gamma(1{-}\gamma), \qquad \frac{V_{Wx} W}{V} = (1{-}\gamma) b,$$

and because the consumption FOC implies[3]

$$\frac{f(W/p(x), V)}{V} = \theta\left(\frac{1}{p(x)} - \delta\right),$$

the HJB reduces to

$$0 = \theta\left(\frac{1}{p(x)} - \delta\right) + (1-\gamma)\left[\mu_0 + \mu_1 x + \frac{\psi}{\theta} b\, \xi(\bar{x} - x) + \frac{1}{2}\sigma_w^2\right]$$

$$-\frac{1}{2}\gamma(1-\gamma)\sigma_w^2 + b\xi(\bar{x} - x) + \frac{1}{2} b^2 \sigma_x^2 + (1-\gamma) b \sigma_w \sigma_x.$$

Using $(1-\gamma)\psi/\theta = \psi(1 - 1/\psi) = \psi - 1$, this becomes

$$0 = \frac{\theta}{p(x)} - \delta\theta + (1-\gamma)(\mu_0 + \mu_1 x) + \psi b \xi(\bar{x} - x) + \frac{1}{2}(1-\gamma)^2 \sigma_w^2 + \frac{1}{2} b^2 \sigma_x^2 + (1-\gamma) b \sigma_w \sigma_x. \tag{2}$$

At this point the continuous-time model differs from the exact discrete-time log-linear benchmark. Since

$$p(x) = \delta^{-\psi} e^{(\psi/\theta)(a+bx)},$$

equation (2) is not affine in x. The exponential-affine guess is therefore not an exact closed

[2]In an exchange economy, μ_W is the total return on the wealth claim, including the dividend yield $1/p(x)$. As a result, the generator of W enters as $V_W W \mu_W$ without a separate $-c$ correction; the consumption choice is absorbed into the aggregator f.

[3]This identity follows from the form of the Epstein-Zin aggregator; see the companion notebook Recursive Utility in Continuous Time for the derivation.

class for the primitive HJB unless $b = 0$ or the model is further specialized.

A standard way forward is to take a local approximation around the steady-state mean $x = \bar{x}$:

$$\frac{1}{p(x)} \approx \frac{1}{\bar{p}} - \frac{\psi}{\theta}\frac{b}{\bar{p}}(x - \bar{x}), \qquad \bar{p} \equiv \delta^{-\psi} e^{(\psi/\theta)(a+b\bar{x})}.$$

Substituting this linearization into (2) and matching the constant and $x - \bar{x}$ coefficients gives two equations for the coefficients a and b. The $x - \bar{x}$ coefficient implies

$$\psi b\left(\xi + \frac{1}{\bar{p}}\right) = (1-\gamma)\mu_1,$$

and the constant term implies

$$\delta\theta = \frac{\theta}{\bar{p}} + (1-\gamma)(\mu_0 + \mu_1\bar{x}) + \frac{1}{2}(1-\gamma)^2\sigma_w^2 + \frac{1}{2}b^2\sigma_x^2 + (1-\gamma)b\sigma_w\sigma_x.$$

Because $\sigma_w = \sigma_c + (\psi/\theta)b\sigma_x$ appears in the constant-term equation, the two matching equations form a jointly nonlinear system in $(b, \bar{p})$ and must be solved simultaneously. Once $\bar{p}$ and b are determined, a follows from the definition of $\bar{p}$:

$$a = \frac{\theta}{\psi}\ln\bar{p} - \theta\ln\delta - b\bar{x}.$$

The quality of the affine approximation improves as b becomes small, which occurs when μ_1 is small (weak state dependence in consumption growth) or when ξ and $\bar{p}^{-1}$ are large relative to $(1-\gamma)\mu_1/\psi$ (fast mean reversion absorbs state variation before it materially affects continuation utility).

Property 3 (Affine Wealth-Consumption Ratio)**.** *Under the local exponential-affine approximation $h(x) = e^{a+bx}$,*

$$p(x) = \delta^{-\psi} e^{(\psi/\theta)(a+bx)},$$

so the wealth-consumption ratio is itself exponential-affine in the state.

Approximate SDF

Once the coefficients a and b are determined, the general SDF from the theory notebook becomes

$$\Lambda_t = \Lambda_0 \exp\left(\int_0^t f_v(C_s, V_s)\, ds\right) \frac{V_W(W_t, x_t)}{V_W(W_0, x_0)}.$$

Since

$$V_W(W_t, x_t) = e^{a+bx_t} W_t^{-\gamma},$$

we have

$$\frac{V_W(W_t, x_t)}{V_W(W_0, x_0)} = e^{b(x_t - x_0)} \left(\frac{W_t}{W_0}\right)^{-\gamma}.$$

In addition, differentiating the Epstein-Zin aggregator with respect to V and using the consumption first-order condition gives[4]

$$f_v(C_t, V_t) = \frac{\theta - 1}{p(x_t)} - \delta\theta.$$

Therefore

$$\Lambda_t = \Lambda_0 \exp\left(-\delta\theta t + (\theta - 1)\int_0^t \frac{ds}{p(x_s)}\right) e^{b(x_t - x_0)} \left(\frac{W_t}{W_0}\right)^{-\gamma}.$$

This is the continuous-time analogue of the discrete-time EZ SDF: one part loads on intertemporal substitution through the reciprocal of the wealth-consumption ratio, another part loads on the state variable through $b(x_t - x_0)$, and the last part is the usual marginal-wealth term $W_t^{-\gamma}$.

Under the local affine approximation

$$\frac{1}{p(x)} \approx \frac{1}{\bar{p}} - \frac{\psi}{\theta}\frac{b}{\bar{p}}(x - \bar{x}),$$

the SDF becomes

$$\Lambda_t \approx \Lambda_0 \exp\left(\left[\frac{\theta - 1}{\bar{p}} - \delta\theta\right] t - (\theta - 1)\frac{\psi}{\theta}\frac{b}{\bar{p}}\int_0^t (x_s - \bar{x})\, ds + b(x_t - x_0)\right)\left(\frac{W_t}{W_0}\right)^{-\gamma}.$$

[4]This formula is derived in the companion notebook Recursive Utility in Continuous Time. As a check: when $\theta = 1$ (time-separable power utility), it reduces to $f_v = -\delta$, consistent with the standard aggregator $f(C, V) = u(C) - \delta V$.

At that point all dependence on the state is explicit, and pricing reduces to evaluating exponential-affine expectations of (W_t, x_t) under the measure induced by this kernel.

The SDF Diffusion

Taking logs gives

$$\ln \Lambda_t = \ln \Lambda_0 - \delta\theta t + (\theta - 1)\int_0^t \frac{ds}{p(x_s)} + b(x_t - x_0) - \gamma \ln\left(\frac{W_t}{W_0}\right).$$

Applying Ito's lemma and using the affine approximation ($q' = b, q'' = 0$), so that

$$d \ln W_t = \left[\mu_0 + \mu_1 x_t + \frac{\psi}{\theta} b\, \xi(\bar{x} - x_t)\right] dt + \sigma_w\, dB_t,$$

and

$$dx_t = \xi(\bar{x} - x_t)\, dt + \sigma_x\, dB_t$$

yields

$$d \ln \Lambda_t = \left[-\delta\theta + \frac{\theta - 1}{p(x_t)} + b\xi(\bar{x} - x_t) - \gamma\left(\mu_0 + \mu_1 x_t + \frac{\psi}{\theta} b\, \xi(\bar{x} - x_t)\right)\right] dt + \left(b\sigma_x - \gamma\sigma_w\right) dB_t.$$

Since

$$\frac{d\Lambda_t}{\Lambda_t} = d \ln \Lambda_t + \frac{1}{2}\left(b\sigma_x - \gamma\sigma_w\right)^2 dt,$$

the SDF diffusion is

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t\, dt - \lambda_t\, dB_t,$$

with market price of risk

$$\lambda_t = \gamma\sigma_w - b\sigma_x.$$

Note that λ_t is constant: the one-factor Gaussian structure forces all state dependence to cancel, a property that does not hold in more general affine EZ specifications. The short rate is

$$r_t = \delta\theta - \frac{\theta - 1}{p(x_t)} - b\xi(\bar{x} - x_t) + \gamma\left(\mu_0 + \mu_1 x_t + \frac{\psi}{\theta} b\, \xi(\bar{x} - x_t)\right) - \frac{1}{2} b^2\sigma_x^2 - \frac{1}{2}\gamma^2\sigma_w^2 + \gamma b \sigma_w \sigma_x.$$

Equivalently,

$$r_t = \delta\theta + \frac{1-\theta}{p(x_t)} + \left(\frac{\gamma\psi}{\theta} - 1\right) b\xi(\bar{x} - x_t) + \gamma(\mu_0 + \mu_1 x_t) - \frac{1}{2}\gamma^2\sigma_w^2 - \frac{1}{2}b^2\sigma_x^2 + \gamma b\sigma_w\sigma_x,$$

where the second line simply collects the terms multiplying $1/p(x_t)$ and $\xi(\bar{x} - x_t)$.

Under the local approximation, $1/p(x_t)$ and $\xi(\bar{x} - x_t)$ are both linear in x_t, so r_t is affine in the state. Together with the constant market price of risk, this reduces bond pricing to a Vasicek-type calculation — the core tractability payoff of the approximation.

This is the continuous-time counterpart of the discrete-time EZ decomposition. The term $\gamma\sigma_w$ is the standard marginal-wealth component of the price of risk, while $-b\sigma_x$ is the extra intertemporal-hedging component generated by recursive utility through state dependence in continuation value.

Property 4 (Approximate SDF and Short Rate). *Under the local affine approximation, the market price of risk is constant:*

$$\lambda_t = \gamma\sigma_w - b\sigma_x,$$

and the short rate is affine in the state:

$$r_t = \delta\theta + \frac{1-\theta}{p(x_t)} + \left(\frac{\gamma\psi}{\theta} - 1\right) b\xi(\bar{x} - x_t) + \gamma(\mu_0 + \mu_1 x_t) - \frac{1}{2}\gamma^2\sigma_w^2 - \frac{1}{2}b^2\sigma_x^2 + \gamma b\sigma_w\sigma_x.$$

References

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