Mean-Variance Analysis with Short-Sales Constraints

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Introduction

The previous notebooks analyzed the frictionless benchmark: with unrestricted portfolio weights, the minimum variance frontier is a single parabola in (σ^2, μ) space, and beta pricing holds globally on it. With a risk-free asset, the same logic yields a single-beta pricing relation with respect to the tangency portfolio.

Imposing **short-sales constraints** ($w_i \geq 0$ for all i) breaks that global structure. The constrained frontier is no longer a single parabola but a collection of parabolic arcs, each corresponding to a different set of held assets.[1] This piecewise structure means that beta pricing holds locally — on each arc separately — but not globally. Assets excluded from a given arc are not priced by its beta relation; instead, their KKT multipliers reveal that they earn weakly less than the arc's pricing formula would predict.

This notebook derives these results. In Markowitz's early work, Markowitz (1952) provides the preference-based motivation for mean-variance choice, while Markowitz (1959) develops the portfolio-selection framework for constrained frontier analysis. The Critical Line Algorithm that traces the constrained frontier numerically is covered in a companion notebook.

[1] Because no short sales implies that feasible portfolios are convex combinations of the original assets, the weight set is the simplex $\{\mathbf{w} : w_i \geq 0,\ \mathbf{w}'\boldsymbol{\iota} = 1\}$, which is compact. Mean and variance are continuous in $\mathbf{w}$, so the investment opportunity set is bounded in (σ, μ) space. In particular, variance is no longer unbounded above, so there is also an upper envelope. Our focus, however, is on the lower envelope, which determines mean-variance efficient portfolios.

Constrained Minimum Variance Frontier

Problem Formulation

The minimum variance portfolio with target return μ and no short sales solves

$$\min_{\mathbf{w}} \; \tfrac{1}{2}\,\mathbf{w}'\mathbf{V}\mathbf{w} \quad \text{s.t.} \quad \mathbf{w}'\mathbf{e} = \mu, \quad \mathbf{w}'\boldsymbol{\iota} = 1, \quad w_i \geq 0 \;\forall\, i.$$

This is a **quadratic program**: a strictly convex objective with linear equality and inequality constraints. Because **V** is positive definite, the solution is unique for every target return $\mu \in [e_{\text{min}}, e_{\text{max}}]$, the range within which the feasible set is non-empty (Boyd and Vandenberghe 2004).

KKT Conditions

The Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\,\mathbf{w}'\mathbf{V}\mathbf{w} - \lambda(\mathbf{w}'\mathbf{e} - \mu) - \gamma(\mathbf{w}'\boldsymbol{\iota} - 1) - {}'\mathbf{w},$$

where λ is the multiplier on the return constraint, γ is the multiplier on the budget constraint, and $\geq \mathbf{0}$ is the vector of multipliers on the non-negativity constraints. Since **V** is positive definite the KKT conditions are necessary and sufficient:

$$\mathbf{V}\mathbf{w} = \lambda\mathbf{e} + \gamma\boldsymbol{\iota} + , \qquad \mathbf{w}'\boldsymbol{\iota} = 1, \qquad \mathbf{w}'\mathbf{e} = \mu,$$

together with complementary slackness: $\nu_i \geq 0$, $w_i \geq 0$, and $\nu_i w_i = 0$ for all i.

Complementary slackness partitions assets into two sets:

- **Active set** $S = \{i : w_i > 0\}$: assets held with positive weight. For these, $\nu_i = 0$ and the stationarity condition reduces to $(\mathbf{V}\mathbf{w})_i = \lambda e_i + \gamma$.
- **Inactive set** $S^c = \{i : w_i = 0\}$: excluded assets. For these, $\nu_i = (\mathbf{V}\mathbf{w})_i - \lambda e_i - \gamma \geq 0$.

Piecewise Parabolic Structure

For a fixed active set S, the optimality conditions are identical to the unconstrained problem restricted to the $|S|$ assets in S. Let $\mathbf{V}_S$, $\mathbf{e}_S$, $\boldsymbol{\iota}_S$ denote the relevant submatrix and subvectors, and define

$$A_S = \boldsymbol{\iota}_S'\mathbf{V}_S^{-1}\mathbf{e}_S, \quad B_S = \mathbf{e}_S'\mathbf{V}_S^{-1}\mathbf{e}_S, \quad C_S = \boldsymbol{\iota}_S'\mathbf{V}_S^{-1}\boldsymbol{\iota}_S, \quad D_S = B_S C_S - A_S^2.$$

The minimum variance on this segment is

$$\sigma^2(\mu) = \frac{1}{D_S}\left(B_S - 2A_S\,\mu + C_S\,\mu^2\right),$$

a parabola in (σ^2, μ) space. On the same segment, the active weights are affine in μ,

$$w_i(\mu) = a_{w,i} + b_{w,i}\mu, \qquad i \in S,$$

and for each excluded asset $j \notin S$ the KKT multiplier is also affine,

$$\nu_j(\mu) = a_{\nu,j} + b_{\nu,j}\mu.$$

The active set remains fixed as μ varies within an interval; at the endpoints, one of these affine conditions hits zero and the active set changes.

The constrained frontier is a **piecewise parabola**, with each piece corresponding to a different active set. The transition points — called **corner portfolios** — are the endpoints of these segments, where an active asset reaches $w_i = 0$ or an excluded asset reaches $\nu_j = 0$. At a corner, one or more assets sit exactly on the boundary between being held and being excluded. Every portfolio on a given segment is a convex combination of its two surrounding corner portfolios.

Asset Pricing Implications

Local Beta Pricing

Within each segment (active set S), the weight vector $\mathbf{w}(\mu)$ traces the unconstrained efficient frontier restricted to S. The beta-pricing result therefore applies to this restricted problem: for any frontier portfolio p on the segment, there is a zero-covariance return μ_z — the intercept of the tangent to the segment's parabola at p — such that every asset $i \in S$ satisfies

$$\mathsf{E}(r_i) = \mu_z + \beta_i \left(\mathsf{E}(r_p) - \mu_z\right).$$

The portfolio with return μ_z holds only assets in S with unconstrained weights and generally lies outside the constrained frontier. Beta pricing holds on each parabolic piece separately, with μ_z shifting at every corner as the active set changes — it does not hold globally across the entire constrained frontier.

Inactive Assets and Pricing Errors

Fix a segment on the efficient branch and a frontier portfolio p on that segment. For active assets $i \in S$, the stationarity condition gives

$$\mathsf{Cov}(r_i, r_p) = (\mathbf{Vw})_i = \lambda e_i + \gamma.$$

Since beta pricing holds exactly on that segment, the constants λ and γ are pinned down by the line

$$e_i = \mu_z + \beta_i \left(\mathsf{E}(r_p) - \mu_z\right), \qquad i \in S.$$

For an inactive asset $j \notin S$, the KKT dual condition implies

$$\mathsf{Cov}(r_j, r_p) = (\mathbf{Vw})_j \geq \lambda e_j + \gamma,$$

with strict inequality whenever the constraint binds strictly. Along the efficient branch we have $\lambda > 0$, so dividing by λ and rearranging yields

$$e_j \leq \mu_z + \beta_j \left(\mathsf{E}(r_p) - \mu_z\right).$$

Thus the local beta formula weakly over-predicts the expected return of any excluded asset:

$$\alpha_j \equiv e_j - \left[\mu_z + \beta_j \left(\mathsf{E}(r_p) - \mu_z\right)\right] \leq 0.$$

Equality holds only at a corner portfolio, when $\nu_j = 0$ and asset j sits exactly on the boundary of the active set.

This sign condition is the mean-variance analogue of the friction-pricing inequalities in He and Modest (1995) and Luttmer (1996). Working in a general stochastic discount factor framework, those papers show that a binding short-sales constraint forces a non-positive pricing error for the constrained asset. The KKT multiplier ν_j plays exactly the role of the constraint's shadow price in their framework — positive when the constraint binds and the pricing error is strict, zero at corners where the constraint is about to become slack. A related equilibrium intuition appears in Hong and Stein (2003): when pessimistic investors cannot short aggressively, prices need not fully incorporate their information, so constraints can generate systematic departures from frictionless benchmark relations. In the present mean-variance setting the mechanism is summarized by the KKT wedge ν_j, but the message is similar: short-sales constraints distort pricing precisely because excluded assets cannot be freely used to restore the unconstrained frontier relation.

Beta Pricing with a Risk-Free Asset

When a risk-free asset at rate r_f is introduced, the zero-covariance term no longer depends on the segment. In the all-risky case, each parabolic piece has its own tangent intercept μ_z, so the beta-pricing relation changes whenever the active set changes. With a risk-free asset, that moving intercept is replaced by the fixed return r_f, because r_f has zero covariance with every risky portfolio. The role previously played by the segment-specific zero-covariance portfolio is now played by the same asset everywhere on the constrained frontier.

Geometrically, the constrained efficient frontier is therefore a ray starting at r_f and tangent to the constrained risky frontier at the tangency portfolio q, the portfolio on the constrained frontier with the highest Sharpe ratio. The tangency portfolio is the risky portfolio relevant for pricing, but it need not contain every asset. Typically it lies in the interior of one parabolic segment, so only the assets in that segment's active set enter the exact beta relation.

Beta pricing holds for assets in the active set S_q of q:

$$\mathrm{E}(r_i) = r_f + \beta_i \left(\mathrm{E}(r_q) - r_f\right), \qquad \beta_i = \frac{\mathrm{Cov}(r_i, r_q)}{\sigma^2(r_q)}, \qquad i \in S_q.$$

For inactive assets $j \notin S_q$ the KKT argument of the previous section still applies: $\nu_j \geq 0$ implies that the constrained beta formula weakly over-predicts expected return, so $\alpha_j \leq 0$. Thus the risk-free asset removes the segment-by-segment variation in the zero-covariance benchmark, but it does not remove the distinction between included and excluded assets. Exact beta pricing holds for the assets that are actually held in the tangency portfolio; excluded assets continue to have weakly negative pricing errors because their non-negativity constraints remain binding. An equilibrium interpretation of this result is deferred to the CAPM notebook.

References

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