Optimal Portfolio and Consumption in a One-Factor Gaussian Model

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Introduction

The Optimal Portfolio and Consumption in Continuous Time notebook derived the general HJB equation, Merton's portfolio decomposition, and the SDF under additive utility. This sequel specializes those results to the most tractable nontrivial state-space example: log consumption growth is Gaussian, but its conditional mean varies with one mean-reverting state variable.

The payoff from this specialization is that the policy functions can be written explicitly in terms of one-dimensional integrals. We obtain three closed-form objects:

1. the **value function**,
2. the **wealth-consumption ratio** W/C,
3. the **optimal risky portfolio**, split into myopic and hedging demands.

The structure is also a useful bridge between the general HJB notebook and the Affine Recursive Utility in Continuous Time notebook: here the solution is exact because additive CRRA utility keeps the SDF linear in log consumption.

A One-Factor Lognormal Economy

Assume aggregate consumption follows

$$d \ln C_t = (\mu_0 + \mu_1 x_t)\, dt + \sigma_c\, dB_t,$$

where the state variable x_t is Ornstein-Uhlenbeck:

$$dx_t = \kappa(\bar{x} - x_t)\,dt + \sigma_x\,dB_t. \tag{1}$$

The same Brownian motion drives both processes, so the economy has one priced shock. A representative investor has CRRA utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0,\ \gamma \neq 1,$$

and solves

$$\max_{\{c_s,\alpha_s\}_{s\geq t}} \mathrm{E}_t\left[\int_t^\infty e^{-\delta(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma}\,ds\right].$$

In equilibrium, the optimal consumption choice equals the endowment process C_t, and wealth is the value of the claim to that consumption stream:

$$W_t = \text{price at } t \text{ of } \{C_{t+s}\}_{s\geq 0}.$$

The SDF from the previous notebook is

$$\Lambda_t = e^{-\delta t}C_t^{-\gamma}.$$

Applying Ito's lemma gives

$$\frac{d\Lambda_t}{\Lambda_t} = -r(x_t)\,dt - \lambda\,dB_t, \qquad \lambda = \gamma\sigma_c,$$

with short rate

$$r(x) = \delta + \gamma(\mu_0 + \mu_1 x) - \frac{1}{2}\gamma^2\sigma_c^2. \tag{2}$$

Thus expected consumption growth, the short rate, and asset prices all move with the same state variable x_t.

Wealth-Consumption Ratio

Let

$$p(x_t) \equiv \frac{W_t}{C_t}$$

denote the wealth-consumption ratio. By definition,

$$p(x_t) = \mathrm{E}_t\left[\int_0^\infty e^{-\delta\tau}\left(\frac{C_{t+\tau}}{C_t}\right)^{1-\gamma} d\tau\right]. \tag{3}$$

Because x_t is Gaussian, the log consumption increment is conditionally normal. Define

$$\ell(\tau) \equiv \frac{1-e^{-\kappa\tau}}{\kappa}, \qquad a \equiv \frac{\mu_1\sigma_x}{\kappa}.$$

The parameter a is the long-run addition to consumption growth volatility from the state variable channel: as $\tau \to \infty$ the diffusion coefficient in the stochastic integral below converges to $\sigma_c + a$, so a measures how much extra consumption risk the state variable contributes once its transient fluctuations have died out.

Integrating (1) implies

$$\Delta c_{t,\tau} \equiv \ln\left(\frac{C_{t+\tau}}{C_t}\right) = \mu_0\tau + \mu_1[\bar{x}\tau + (x_t - \bar{x})\ell(\tau)] + \int_t^{t+\tau}\left[\sigma_c + a\left(1 - e^{-\kappa(t+\tau-u)}\right)\right] dB_u.$$

Hence $\Delta c_{t,\tau} \mid x_t$ is normal with mean

$$m_c(\tau, x_t) = \mu_0\tau + \mu_1[\bar{x}\tau + (x_t - \bar{x})\ell(\tau)],$$

and variance

$$v_c(\tau) = (\sigma_c + a)^2\tau - 2a(\sigma_c + a)\ell(\tau) + \frac{a^2}{2\kappa}\left(1 - e^{-2\kappa\tau}\right).$$

Using the lognormal moment formula in (3) gives the exact ratio

$$p(x) = \int_0^\infty \exp\left(A(\tau) + B(\tau)x\right) d\tau, \tag{4}$$

where

$$B(\tau) = (1-\gamma)\mu_1 \ell(\tau),$$

and

$$A(\tau) = -\delta\tau + (1-\gamma)\big[\mu_0\tau + \mu_1\bar{x}\big(\tau - \ell(\tau)\big)\big] + \frac{1}{2}(1-\gamma)^2 v_c(\tau).$$

The integral converges when the long-run exponent is negative:

$$-\delta + (1-\gamma)(\mu_0 + \mu_1\bar{x}) + \frac{1}{2}(1-\gamma)^2\left(\sigma_c + \frac{\mu_1\sigma_x}{\kappa}\right)^2 < 0. \tag{5}$$

Economically, this condition ensures that the present value of the future consumption stream is finite; if it fails, the wealth claim has infinite value and no consumption optimum exists.

Equation (4) is the continuous-time analogue of the wealth-consumption ratio in the discrete-time lognormal consumption model. The state enters only through the affine loading $B(\tau)x$; the remaining term $A(\tau)$ depends only on horizon τ.

Property 1 (Exact Wealth-Consumption Ratio)**.** *In the one-factor lognormal economy,*

$$p(x_t) = \frac{W_t}{C_t} = \int_0^\infty \exp\big(A(\tau) + B(\tau)x_t\big)\,d\tau.$$

provided the transversality condition (5) holds.

Value Function and Optimal Consumption

Homotheticity suggests the usual guess

$$V(W,x) = \frac{W^{1-\gamma}}{1-\gamma}h(x).$$

The consumption FOC from the previous notebook implies

$$u'(c_t) = V_W(W_t, x_t).$$

Specializing to equilibrium, we have $c_t = C_t$. For CRRA utility and the homothetic value function,

$$u'(C_t) = C_t^{-\gamma}, \qquad V_W(W_t, x_t) = W_t^{-\gamma} h(x_t),$$

so

$$\left(\frac{W_t}{C_t}\right)^{\gamma} = h(x_t).$$

Since $W_t = C_t p(x_t)$, it follows that

$$h(x) = p(x)^{\gamma}.$$

Therefore the exact value function is

$$V(W, x) = \frac{W^{1-\gamma}}{1-\gamma} p(x)^{\gamma}, \tag{6}$$

and the optimal wealth-consumption ratio is

$$\frac{W_t}{C_t} = p(x_t). \tag{7}$$

Portfolio and Wealth Dynamics

Suppose there is one traded risky asset spanning the Brownian shock:

$$\frac{dS_t}{S_t} = \big(r(x_t) + \sigma_S \lambda\big)\, dt + \sigma_S\, dB_t,$$

with $\lambda = \gamma \sigma_c$ from the SDF above. Let α_t denote the share of wealth invested in this asset.

From the HJB first-order condition derived in the companion notebook, the general one-state Merton rule gives

$$\alpha_t^* = \frac{\lambda}{\gamma \sigma_S} + \frac{\sigma_x}{\gamma \sigma_S} \frac{\partial \ln V_W(W_t, x_t)}{\partial x}.$$

Since

$$V_W(W, x) = W^{-\gamma} p(x)^{\gamma},$$

we have

$$\frac{\partial \ln V_W(W,x)}{\partial x} = \gamma \frac{p'(x)}{p(x)},$$

so the optimal portfolio simplifies to

$$\alpha_t^* = \frac{\sigma_c}{\sigma_S} + \frac{\sigma_x}{\sigma_S}\frac{p'(x_t)}{p(x_t)}. \tag{8}$$

Differentiating (4) under the integral sign yields

$$p'(x) = \int_0^\infty B(\tau)\exp\big(A(\tau) + B(\tau)x\big)\,d\tau,$$

and therefore

$$\alpha_t^* = \frac{\sigma_c}{\sigma_S} + \frac{\sigma_x}{\sigma_S}\frac{\int_0^\infty B(\tau)\exp\big(A(\tau)+B(\tau)x_t\big)\,d\tau}{\int_0^\infty \exp\big(A(\tau)+B(\tau)x_t\big)\,d\tau}. \tag{9}$$

Property 2 (Closed-Form Portfolio Rule)**.** *The optimal risky share is*

$$\alpha_t^* = \underbrace{\frac{\sigma_c}{\sigma_S}}_{\text{myopic demand}} + \underbrace{\frac{\sigma_x}{\sigma_S}\frac{p'(x_t)}{p(x_t)}}_{\text{hedging demand}}.$$

The first term is the standard Merton myopic demand. The second term is the hedging demand: when the state variable raises the wealth-consumption ratio, so that $p'(x_t) > 0$, the investor takes extra risky exposure to hedge changes in future investment opportunities.

As a consistency check — and to express the portfolio rule in terms of aggregate quantities — we derive the dynamics of equilibrium wealth directly from $W_t = C_t p(x_t)$. This will also yield an equivalent, more compact expression for α_t^*.

Since equilibrium wealth satisfies $W_t = C_t p(x_t)$, we have

$$\ln W_t = \ln C_t + \ln p(x_t).$$

Applying Ito's lemma and using $d\ln C_t = (\mu_0 + \mu_1 x_t)\,dt + \sigma_c\,dB_t$ together with

$$dx_t = \kappa(\bar{x} - x_t)\,dt + \sigma_x\,dB_t$$

implies

$$d\ln W_t = \left[\mu_0 + \mu_1 x_t + \kappa(\bar{x} - x_t)\frac{p'(x_t)}{p(x_t)} + \frac{1}{2}\sigma_x^2\left(\frac{p''(x_t)}{p(x_t)} - \left(\frac{p'(x_t)}{p(x_t)}\right)^2\right)\right]dt$$

$$+\left[\sigma_c + \sigma_x\frac{p'(x_t)}{p(x_t)}\right]dB_t.$$

Therefore

$$\frac{dW_t}{W_t} = \left[\mu_0 + \mu_1 x_t + \kappa(\bar{x} - x_t)\frac{p'(x_t)}{p(x_t)} + \frac{1}{2}\sigma_x^2\left(\frac{p''(x_t)}{p(x_t)} - \left(\frac{p'(x_t)}{p(x_t)}\right)^2\right) + \frac{1}{2}\left(\sigma_c + \sigma_x\frac{p'(x_t)}{p(x_t)}\right)^2\right]dt$$

$$+\left(\sigma_c + \sigma_x\frac{p'(x_t)}{p(x_t)}\right)dB_t.$$

We record this result in the following property.

Property 3 (Wealth Dynamics)**.** *The equilibrium wealth claim satisfies*

$$\frac{dW_t}{W_t} = \mu_W(x_t)\,dt + \sigma_W(x_t)\,dB_t,$$

where

$$\mu_W(x) \equiv \mu_0 + \mu_1 x + \kappa(\bar{x} - x)\frac{p'(x)}{p(x)} + \frac{1}{2}\sigma_x^2\left(\frac{p''(x)}{p(x)} - \left(\frac{p'(x)}{p(x)}\right)^2\right) + \frac{1}{2}\left(\sigma_c + \sigma_x\frac{p'(x)}{p(x)}\right)^2,$$

and

$$\sigma_W(x) = \sigma_c + \sigma_x\frac{p'(x)}{p(x)}.$$

The portfolio rule can therefore be written equivalently as

$$\alpha_t^* = \frac{\sigma_W(x_t)}{\sigma_S}.$$

The investor chooses exactly the exposure needed to reproduce the volatility of the aggregate wealth claim. If the traded risky asset is itself the claim to aggregate consumption, then $\sigma_S =$

$\sigma_W(x_t)$ and the representative agent simply holds the entire wealth portfolio: $\alpha_t^* = 1$. Equation (8) is more useful when the primitive traded asset spans the same shock but differs from the aggregate wealth claim.

Finally, the SDF pins down the expected total return on the wealth claim, which includes the dividend yield $C_t/W_t = 1/p(x_t)$. Using the SDF pricing equation, the total expected return on wealth is

$$\mu_W(x) + \frac{1}{p(x)} = r(x) + \gamma\sigma_c\sigma_W(x),$$

so the risk premium is

$$\mu_W(x) + \frac{1}{p(x)} - r(x) = \gamma\sigma_c\sigma_W(x).$$

Special Cases

When $\mu_1 = 0$, expected consumption growth is constant. Then $B(\tau) = 0$, so $p(x)$ is constant, $p'(x) = 0$, and the state variable drops out of the solution. The model collapses to the standard Merton benchmark:

$$\frac{W_t}{C_t} = \text{constant}, \qquad \alpha_t^* = \frac{\sigma_c}{\sigma_S}.$$

There is no hedging demand because investment opportunities no longer vary over time.

When $\gamma = 1$, the CRRA formulas above become singular because log utility must be handled separately. Setting $\gamma = 1$ directly in the integrand, $B(\tau) = (1-1)\mu_1\ell(\tau) = 0$ and $A(\tau) = -\delta\tau$, so the integrand reduces to $e^{-\delta\tau}$ and

$$p(x) = \int_0^\infty e^{-\delta\tau}\, d\tau = \frac{1}{\delta}.$$

The wealth-consumption ratio is constant and equal to $1/\delta$, independent of the state variable, so there is again no hedging demand.

Conclusion

This one-factor Gaussian specification is the simplest exact state-dependent solution to the continuous-time consumption-portfolio problem with additive utility. The main objects all reduce to the wealth-consumption ratio $p(x)$:

$$V(W,x) = \frac{W^{1-\gamma}}{1-\gamma} p(x)^{\gamma}, \qquad \frac{W}{C} = p(x), \qquad \alpha^* = \frac{\sigma_c}{\sigma_S} + \frac{\sigma_x}{\sigma_S}\frac{p'(x)}{p(x)}.$$

The state variable matters only through how it changes the present value of future consumption growth. That is exactly the hedging channel in Merton's ICAPM: when the state changes the entire future path of investment opportunities, optimal portfolio choice depends not only on today's Sharpe ratio, but also on how wealth responds to that state.