

The Heston Model

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The Model

The Heston (1993) model is one of the most cited papers in finance, and for good reason: it provides a tractable framework for pricing options under stochastic volatility. In the Black-Scholes world, the volatility of returns is constant, which is at odds with the empirical observation that implied volatilities vary across strikes and maturities—the so-called volatility smile or skew. The Heston model addresses this by allowing the instantaneous variance of returns to evolve as a mean-reverting stochastic process.

Specifically, the stock price S and its instantaneous variance v follow the joint diffusion

$$\begin{aligned}\frac{dS}{S} &= \mu dt + \sqrt{v} dB_1, \\ dv &= \kappa(\theta - v) dt + \sigma\sqrt{v} dB_2,\end{aligned}$$

where B_1 and B_2 are correlated Brownian motions satisfying $dB_1 dB_2 = \rho dt$. The variance process follows a **Cox-Ingersoll-Ross (CIR)** process, which keeps the variance non-negative and strictly positive whenever the Feller condition $2\kappa\theta > \sigma^2$ holds. The parameter $\kappa > 0$ governs the speed of mean-reversion of v toward its long-run mean θ , and $\sigma > 0$ is the **volatility of volatility**. The correlation ρ between the two Brownian motions captures the empirically documented **leverage effect**: in equity markets $\rho < 0$, so that price declines tend to coincide with increases in variance.

The Stochastic Discount Factor

A natural SDF specification in this two-factor setting is

$$\frac{d\Lambda}{\Lambda} = -r dt - \frac{\lambda_1}{\sqrt{v}} dB_1 - \frac{\lambda_2}{\sqrt{v}} dB_2.$$

The market prices of risk are λ_1/\sqrt{v} for the stock-price risk factor and λ_2/\sqrt{v} for the variance risk factor. Scaling by $1/\sqrt{v}$ keeps the risk prices bounded as $v \rightarrow 0$ and produces an affine structure under the risk-neutral measure.

By Girsanov's theorem, define the risk-neutral measure P^* via $dP^*/dP = \Lambda_T/\Lambda_0$, so that

$$dB_1^* = dB_1 + \frac{\lambda_1}{\sqrt{v}} dt, \quad dB_2^* = dB_2 + \frac{\lambda_2}{\sqrt{v}} dt$$

define Brownian motions under P^* with $dB_1^* dB_2^* = \rho dt$. Substituting $dB_i = dB_i^* - \lambda_i/\sqrt{v} dt$ into the physical dynamics gives the risk-neutral dynamics of the stock and variance:

$$\begin{aligned} \frac{dS}{S} &= (r - q) dt + \sqrt{v} dB_1^*, \\ dv &= \kappa(\theta^* - v) dt + \sigma\sqrt{v} dB_2^*, \end{aligned}$$

where $\mu - \lambda_1 = r - q$ is the stock's equilibrium risk-premium condition, and $\theta^* \equiv \theta - \sigma\lambda_2/\kappa$ is the risk-neutral long-run variance mean. Under P^* , every derivative on S earns exactly the risk-free rate in expectation, $E^*(dF) = rF dt$.

The Partial Differential Equation

Consider a European-style derivative with price $F(S, v, T)$ written on the stock S and expiring at time T . Under the risk-neutral measure P^* defined by the SDF, every derivative earns exactly the risk-free rate in expectation, $E^*(dF) = rF dt$. The risk-neutral dynamics of (S, v) are

$$\begin{aligned} \frac{dS}{S} &= (r - q) dt + \sqrt{v} dB_1^*, \\ dv &= \kappa(\theta^* - v) dt + \sigma\sqrt{v} dB_2^*, \end{aligned}$$

with $dB_1^* dB_2^* = \rho dt$. Applying Ito's lemma to $F(S, v, T)$ under P^* :

$$\begin{aligned} dF &= F_S dS + F_v dv + \frac{1}{2}F_{SS}(dS)^2 + \frac{1}{2}F_{vv}(dv)^2 + F_{Sv}(dS)(dv) - F_T dt \\ &= \left((r - q)SF_S + \kappa(\theta^* - v)F_v + \frac{1}{2}vS^2F_{SS} + \frac{1}{2}\sigma^2vF_{vv} + \sigma v\rho SF_{Sv} - F_T \right) dt \\ &\quad + F_S S\sqrt{v} dB_1^* + F_v \sigma\sqrt{v} dB_2^*. \end{aligned}$$

Setting the drift equal to rF yields the **Heston PDE**:

$$(r - q)SF_S + \kappa(\theta^* - v)F_v + \frac{1}{2}vS^2F_{SS} + \frac{1}{2}\sigma^2vF_{vv} + \sigma v\rho SF_{Sv} - F_T - rF = 0. \quad (1)$$

This PDE, subject to the terminal condition $F(S, v, 0) = f(S)$ for the specific payoff function f , fully characterizes the price of any European-style derivative written on S in the Heston model.

For most payoffs—including vanilla calls and puts—equation (1) cannot be solved in closed form directly. This was Heston's key insight: rather than solving the PDE for the price itself, one can solve it for the **characteristic function** of the log stock price, which is available in closed form and can then be used to recover option prices via Fourier inversion.

Option Pricing

The characteristic function of a random variable X is defined as $f_X(\phi) = E(e^{i\phi X})$, where $i = \sqrt{-1}$. Knowing the characteristic function is equivalent to knowing the full distribution of X , and the CDF can be recovered via the **Gil-Pelaez inversion formula**. For option pricing, if we know the characteristic function of $\ln S_T$, the in-the-money probabilities can be computed by fast numerical integration.

The price of a European call with strike K and maturity T is

$$C = Se^{-qT}P_1 - Ke^{-rT}P_2,$$

where P_1 and P_2 are risk-adjusted probabilities that the option expires in the money, given by

$$P_1 = \frac{1}{2} + \frac{1}{\pi F} \int_0^\infty \operatorname{Re} \left(\frac{e^{i\phi \ln K} f(\phi - i, 0)}{i\phi} \right) d\phi,$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left(\frac{e^{i\phi \ln K} f(\phi, 0)}{i\phi} \right) d\phi,$$

where $F = S e^{(r-q)T}$ is the forward price and $f(\phi, \varphi)$ is the joint characteristic function of $\ln S(T)$ and $v(T)$. The probability P_2 is the risk-neutral probability that $S_T > K$, while P_1 is the corresponding probability under the stock-numeraire measure. Unlike Heston (1993), who used two distinct characteristic functions, the formulas above use a single characteristic function evaluated at different arguments by switching to the forward measure.

Define $x = \ln S$ so that the risk-neutral dynamics become

$$dx = (r - q) dt + \sqrt{v} dB_1^*,$$

$$dv = \kappa(\theta^* - v) dt + \sigma\sqrt{v} dB_2^*,$$

where B_1^* and B_2^* are correlated Brownian motions under P^* with $dB_1^* dB_2^* = \rho dt$. The joint characteristic function of $x(T)$ and $v(T)$,

$$f(\phi, \varphi) = E^* \left(e^{i\phi x(T) + i\varphi v(T)} \right),$$

satisfies the same PDE (1) with terminal condition $f(\phi, \varphi) = e^{i\phi x + i\varphi v}$ at $T = 0$. Because the model is affine, the characteristic function has a closed-form solution:

$$f(\phi, \varphi) = \exp \left(i(r - q)\phi T + \left(\delta - \frac{2\gamma}{\sigma^2} \right) \kappa\theta^* T + i\phi x + \delta v \right)$$

$$\times \left(e^{-\gamma T} - \frac{\sigma^2}{2\gamma} (1 - e^{-\gamma T}) (i\varphi - \delta) \right)^{-2\kappa\theta^*/\sigma^2}$$

$$\times \exp \left(\frac{v(i\varphi - \delta)}{e^{-\gamma T} - \frac{\sigma^2}{2\gamma} (1 - e^{-\gamma T}) (i\varphi - \delta)} \right),$$

where

$$\gamma = \sqrt{\kappa^2 + (1 - \rho^2)\sigma^2\phi^2 + i(\sigma - 2\kappa\rho)\sigma\phi},$$
$$\delta = \frac{\kappa + \gamma - i\rho\sigma\phi}{\sigma^2}.$$

The expectation is taken under the risk-neutral measure given current values of $x = \ln S$ and v . To price a call option, one only needs $f(\phi, 0)$.

Numerical Implementation

Implementing the Heston formula reliably requires care. The most important consideration is to write every term in the characteristic function using $e^{-\gamma T}$ rather than $e^{\gamma T}$. The complex square root γ has branches that can cause sign flips as ϕ varies, and the formulation with $e^{-\gamma T}$ — which decays for large T — avoids the exponential blow-up that plagues the equivalent $e^{\gamma T}$ formulation. Failing to observe this leads to discontinuities in the integrand and wildly incorrect option prices.

A reliable way to verify the implementation is to exploit the moment-generating property of the characteristic function. Generate a large number of Monte Carlo paths for $x(T)$ and $v(T)$ under P^* , then check that the sample average of $e^{i\phi x(T) + i\varphi v(T)}$ matches the closed-form formula for several values of ϕ and φ . This test should be performed across a range of parameter values to expose numerical instabilities. The same simulation can also verify the final option prices by computing the average of discounted payoffs and comparing to the Fourier-inversion formula.

References

Heston, Steven L. 1993. “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options.” *Review of Financial Studies* 6 (2): 327–43.