The Geometry of the Payoff Space

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In this notebook I describe the mathematical structure of the payoff space that we will use to characterize the space of traded payoffs and stochastic discount factors. Even though the results described in this notebook apply to infinite dimensional Hilbert spaces, we will restrict our attention to the study of finite dimensional Euclidean spaces.

Probability Structure

Uncertainty is represented by a finite set $\mathcal{S} = \{1, \dots, S\}$ of states, defining a finite probability space $(\mathcal{S}, \pi)$. The set of all random variables defined in $\mathcal{S}$ is denoted by L and is called the **payoff space**. Thus, for any $x \in L$ we have that the vector $(x(1), x(2), \dots, x(S)) \in \mathbb{R}^S$ defines all the possible payoffs in each state, and the probability of getting a payoff in a particular state is given by $\Pr(x = x(s)) = \pi(s)$ for all $s \in \mathcal{S}$. We assume throughout that $\pi(s) > 0$ for all $s \in \mathcal{S}$, that is, we will not consider possible outcomes that happen with probability zero.

The payoff space is clearly a linear vector space since for any $x, y \in L$ and $\alpha, \beta \in \mathbb{R}$ we have that $\alpha x + \beta y \in L$. We endow the payoff space with an inner product $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{R}$ defined such that for any $x, y \in L$, we have that

$$\langle x, y \rangle = \mathrm{E}(xy) = \sum_{s=1}^{S} \pi(s) x(s) y(s).$$

In finite-dimensional spaces, we can use the inner product to define the Euclidean norm $\|\cdot\| : L \to \mathbb{R}^+$ for all $x \in L$ as

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Clearly, $\|x\| = 0 \Leftrightarrow x = 0$. The second moment of x, defined as $\|x\|^2 = \mathrm{E}(x^2)$, plays an important role since it allows us to assess the *convergence* of a sequence of payoffs towards a given payoff.

Note that the inner product can be related to standard statistical moments. For any $x, y \in L$, we have

$$\langle x, y \rangle = \mathsf{E}(xy) = \mathsf{Cov}(x, y) + \mathsf{E}(x)\,\mathsf{E}(y),$$

where $\mathsf{Cov}(x, y) = \mathsf{E}[(x - \mathsf{E}(x))(y - \mathsf{E}(y))]$ denotes the covariance between x and y. In particular, $\|x\|^2 = \mathsf{V}(x) + [\mathsf{E}(x)]^2$, where $\mathsf{V}(x) = \mathsf{Cov}(x, x)$ is the variance of x.

Since L consists of all random variables on $\mathcal{S}$, it is a vector space of dimension S. A natural basis is provided by the *Arrow-Debreu securities* $\{e_1, e_2, \dots, e_S\}$, where

$$e_s(i) = \begin{cases} 1 & \text{if } i = s, \\ 0 & \text{otherwise,} \end{cases}$$

for each $s \in \mathcal{S}$. The security e_s pays one unit in state s and zero in every other state. Any payoff $x \in L$ can be expressed in this basis as

$$x = \sum_{s=1}^{S} x(s)\, e_s,$$

since in state i the right-hand side evaluates to $x(i)$. Under the inner product defined above, the Arrow-Debreu securities are mutually orthogonal: for $s \neq t$, the product $e_s e_t$ is zero in every state, so $\langle e_s, e_t \rangle = \mathsf{E}(e_s e_t) = 0$. Moreover, $\langle e_s, e_s \rangle = \mathsf{E}(e_s^2) = \pi(s)$, so the Arrow-Debreu securities form an *orthogonal*—though not orthonormal—basis for L.

Projections

Given $x, y \in L$, consider the vectors $y_x = \alpha x$ and $z = y - y_x$. We say that y_x is the projection of y on the subspace generated by $\{x\}$ if the norm of z is minimal. To obtain the projection, we need to compute the α that minimizes $\|z\|^2 = \|y - \alpha x\|^2 = \mathsf{E}[(y - \alpha x)^2]$. The first-order condition of this problem is:

$$0 = \mathsf{E}[(y - \alpha x)x] = \langle y - \alpha x, x \rangle = \langle z, x \rangle,$$

which implies that $\alpha = \dfrac{\langle x, y \rangle}{\langle x, x \rangle}$ and $\langle z, y_x \rangle = 0$.

We say that two vectors $x, y \in L$ are orthogonal if their inner product is equal to zero. Thus, we have that $y_x \perp z$, implying that the vector y can be decomposed into two orthogonal components. Indeed, we have that

$$\|y\|^2 = \|z + y_x\|^2 = \|z\|^2 + 2\langle z, y_x\rangle + \|y_x\|^2 = \|z\|^2 + \|y_x\|^2,$$

which is a generalization of the classical Pythagorean theorem.

Property 1 (Orthogonal Decomposition). *Given $x, y \in L$, the projection of y on the subspace generated by $\{x\}$ is given by $y_x = \dfrac{\langle x, y\rangle}{\langle x, x\rangle}x$. The vector $z = y - y_x$ is orthogonal to y_x, implying that*

$$\|y\|^2 = \|z\|^2 + \|y_x\|^2. \tag{1}$$

Equation (1) implies that $\|y\|^2 \geq \|y_x\|^2$, with equality occurring whenever y is proportional to x. Therefore, we have that

$$\|y\|^2 \geq \|y_x\|^2 = \left\|\frac{\langle x, y\rangle}{\langle x, x\rangle}x\right\|^2 = \frac{\langle x, y\rangle^2}{\|x\|^2}.$$

The previous expression is known as the Cauchy-Schwartz inequality and is fundamental in the study of Euclidean vector spaces.

Property 2 (Cauchy-Schwartz Inequality). *Given $x, y \in L$ we have that*

$$|\langle x, y\rangle| \leq \|x\|\|y\|. \tag{2}$$

The Cauchy-Schwartz inequality implies that, for any two nonzero $x, y \in L$, the ratio $\dfrac{\langle x, y\rangle}{\|x\|\|y\|}$ lies in $[-1, 1]$ and can be interpreted as the cosine of the angle $\theta \in [0, \pi]$ between x and y. Specifically, $|\langle x, y\rangle| = \|x\|\|y\|$ if and only if y is proportional to x, i.e., the two vectors are collinear. When $\mathrm{E}(x) = \mathrm{E}(y) = 0$, the inner product reduces to $\langle x, y\rangle = \mathrm{Cov}(x, y)$, and the ratio $\dfrac{\langle x, y\rangle}{\|x\|\|y\|}$ coincides with the *correlation coefficient* between x and y.

Projection onto a Subspace

The projection result established in Property 1 extends naturally to higher-dimensional subspaces. Let $M \subseteq L$ be a subspace with linearly independent basis $\{x_1, x_2, \ldots, x_N\}$, and collect the basis elements into the random vector $\mathbf{x} = (x_1, \ldots, x_N)'$. We say that $y_M \in M$ is the *projection* of $y \in L$ onto M if it minimizes $\|y - z\|$ over all $z \in M$.

Since any $z \in M$ takes the form $\mathbf{a}'\mathbf{x}$ for some $\mathbf{a} \in \mathbb{R}^N$, the projection solves

$$\min_{\mathbf{a} \in \mathbb{R}^N} \|y - \mathbf{a}'\mathbf{x}\|^2 = \min_{\mathbf{a} \in \mathbb{R}^N} \mathrm{E}\left[(y - \mathbf{a}'\mathbf{x})^2\right].$$

The first-order condition with respect to $\mathbf{a}$ yields

$$\mathrm{E}\left[(y - \mathbf{a}'\mathbf{x})\mathbf{x}\right] = \mathrm{E}(y\mathbf{x}) - \mathrm{E}(\mathbf{x}\mathbf{x}')\mathbf{a} = \mathbf{0}.$$

Provided the Gram matrix $G = \mathrm{E}(\mathbf{x}\mathbf{x}')$ is invertible, the unique minimizer is $\mathbf{a} = G^{-1}\,\mathrm{E}(y\mathbf{x})$, so

$$y_M = \mathbf{a}'\mathbf{x} = \mathrm{E}(y\mathbf{x})' G^{-1}\mathbf{x}.$$

The residual $z = y - y_M$ satisfies $\langle z, x_i \rangle = 0$ for every $i = 1, \ldots, N$. By linearity, this extends to $\langle z, x \rangle = 0$ for every $x \in M$, and the Pythagorean identity becomes $\|y\|^2 = \|y_M\|^2 + \|z\|^2$.

Property 3 (Projection onto a Subspace)**.** *Let $M \subseteq L$ be a subspace with linearly independent basis $\{x_1, \ldots, x_N\}$, let $\mathbf{x} = (x_1, \ldots, x_N)'$, and assume the Gram matrix $G = \mathrm{E}(\mathbf{x}\mathbf{x}')$ is invertible. For any $y \in L$, the projection onto M is*

$$y_M = \mathrm{E}(y\mathbf{x})' G^{-1}\mathbf{x}.$$

The residual $z = y - y_M$ satisfies $\langle z, x \rangle = 0$ for all $x \in M$, and $\|y\|^2 = \|y_M\|^2 + \|z\|^2$.

The set of all payoffs in L that are orthogonal to M is called the *orthogonal complement* of M:

$$M^\perp = \{z \in L : \langle z, x \rangle = 0 \text{ for all } x \in M\}.$$

Since $M^\perp$ is closed under addition and scalar multiplication, it is itself a subspace of L. The projection theorem implies that every $y \in L$ admits the unique decomposition $y = y_M + z$

with $y_M \in M$ and $z \in M^\perp$, so $M \cap M^\perp = \{0\}$ and

$$L = M \oplus M^\perp.$$

This *orthogonal direct sum* decomposition is the key geometric fact underlying the structure of stochastic discount factors: as we will see in the next notebook, any valid SDF m decomposes as $m = x^* + e$, where $x^* \in X$ is the projection of m onto the subspace of traded payoffs and $e \in X^\perp$ is the orthogonal residual.

Linear Functionals

A central object in asset pricing is a *pricing functional* that assigns a price to every traded payoff. Under the law of one price, such a functional must be linear: the price of a portfolio equals the sum of the prices of its components. Understanding when and how linear functionals can be represented will therefore be essential for characterizing stochastic discount factors.

Given $x, y \in L$ and $\alpha, \beta \in \mathbb{R}$, a linear functional $f : L \to \mathbb{R}$ satisfies

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

We say that the linear functional $f : L \to \mathbb{R}$ is bounded if

$$|f(x)| \leq M\|x\|$$

for all $x \in L$. In other words, the absolute value of the functional cannot grow infinitely for a finite x. A bounded linear functional is also called a continuous linear functional. The smallest M for which this inequality remains true is called the norm of f, i.e.,

$$\|f\| = \inf\{M : |f(x)| \leq M\|x\|, \text{ for all } x \in L\}.$$

For a given $m \in L$ and any $x \in L$, the functional

$$f(x) = \langle m, x \rangle = \mathsf{E}(mx) = \sum_{s=1}^{S} \pi(s) m(s) x(s)$$

is linear since

$$f(\alpha x + \beta y) = \mathsf{E}(m(\alpha x + \beta y)) = \alpha\, \mathsf{E}(mx) + \beta\, \mathsf{E}(my) = \alpha f(x) + \beta f(y).$$

Furthermore, the Cauchy-Schwartz inequality implies that

$$|f(x)| = |\langle m, x \rangle| \leq \|m\| \|x\|,$$

showing that the linear functional f is bounded and hence continuous. Since the previous inequality is an equality whenever x is proportional to m, we have that $\|m\|$ is the smallest bound of f, showing that $\|f\| = \|m\|$.

Conversely, consider a linear functional $f : L \to \mathbb{R}$. The set $K = \{x \in L : f(x) = 0\}$ describes a hyperplane that can be described by a normal vector z. Thus, $\langle x, z \rangle = 0$ for all $x \in K$. Without loss of generality, assume that z has been appropriately scaled so that $f(z) = 1$.

Given any $x \in L$, we have that $x - f(x)z \in K$ since $f(x - f(x)z) = f(x) - f(x)f(z) = 0$. Moreover, $z \perp K$, implying that

$$0 = \langle x - f(x)z, z \rangle = \langle x, z \rangle - f(x)\langle z, z \rangle.$$

The previous expression implies that

$$f(x) = \frac{\langle x, z \rangle}{\langle z, z \rangle} = \langle x, m \rangle,$$

where $m = \dfrac{z}{\|z\|^2}$. The previous analysis is an important result known as the *Riesz representation theorem*.

Property 4 (Riesz Representation Theorem). *If $f : L \to \mathbb{R}$ is a bounded linear functional, there exists a unique vector $m \in L$ such that for all $x \in L$, $f(x) = \langle m, x \rangle$. Furthermore, we have*

$\|f\| = \|m\|$ *and every* m *determines a unique bounded linear functional.*

The Riesz representation theorem has a direct and important consequence for asset pricing. Suppose that $p : L \to \mathbb{R}$ is a linear pricing functional satisfying the law of one price. Then the theorem guarantees the existence of a unique $m \in L$ such that

$$p(x) = \langle m, x \rangle = \mathrm{E}(mx)$$

for every payoff $x \in L$. The random variable m is called a *stochastic discount factor*, and characterizing its properties—positivity, uniqueness, and variance bounds—is the central task of asset pricing theory.