The Fundamental Theorem of Asset Pricing with d Risky Assets

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In the stochastic discount factor notebook we proved the SDF existence theorem in a finite state space $\mathbb{R}^S$ using a direct separation argument. That proof exploits two finiteness assumptions simultaneously: a finite number of states ($|\Omega| = S$) and a finite number of risky assets (d assets with strategies in $\mathbb{R}^d$).

This notebook removes the finite-state assumption. We work on an arbitrary probability space $(\Omega, \mathcal{F}, P)$ with payoffs in $L^p = L^p(\Omega, \mathcal{F}, P)$ for any $p \in [1, \infty)$. The strategy space remains finite-dimensional: d risky assets with deterministic portfolio weights $\theta \in \mathbb{R}^d$. This finite dimensionality is the key structural feature: **the Bolzano-Weierstrass theorem implies that simple no-arbitrage (NA) is sufficient** for the existence of a risk-neutral measure, without strengthening to no free lunch (NFL). The companion notebook FTAP: Infinite-Dimensional Strategies treats the harder case where the strategy space itself is infinite-dimensional, and NA must be strengthened to NFL.

The L^p Setting

Let $p \in [1, \infty)$ and let q denote the conjugate exponent defined by $1/p + 1/q = 1$ (with $q = \infty$ when $p = 1$). We work in $L^p = L^p(\Omega, \mathcal{F}, P)$, the space of random variables with finite p-th moment, equipped with the norm $\|x\|_p = \mathrm{E}(|x|^p)^{1/p}$. By the Riesz representation theorem, the dual space satisfies $(L^p)^* = L^q$.

The positive cone $L^p_+ = \{x \in L^p : x \geq 0 \text{ a.s.}\}$ has empty interior for all $p \in [1, \infty)$ on a non-atomic probability space. This rules out a global separation argument of the type used in $\mathbb{R}^S$, but the finite dimensionality of $\mathbb{R}^d$ provides an alternative route.

The One-Period Model

There are d risky assets with price changes $\Delta S = (\Delta S^1, \dots, \Delta S^d) \in (L^p)^d$ and a risk-free numeraire with return normalized to zero. A trading strategy is a vector $\theta \in \mathbb{R}^d$ of portfolio weights, yielding terminal payoff $\theta \cdot \Delta S := \sum_{i=1}^d \theta^i \Delta S^i \in L^p$.

Define the *cone of claims super-replicable at non-positive cost*:

$$A = \{y \in L^p : y \leq \theta \cdot \Delta S \text{ a.s. for some } \theta \in \mathbb{R}^d\},$$

and let $\bar{A}$ denote its closure in L^p. Then $A = R - L^p_+$ where $R = \{\theta \cdot \Delta S : \theta \in \mathbb{R}^d\}$, and $-L^p_+ \subseteq A$ (take $\theta = 0$).

No-Arbitrage Conditions

Assumption 1 (No Arbitrage)**.**

$$A \cap L^p_+ = \{0\}.$$

Assumption 2 (No Free Lunch)**.**

$$\bar{A} \cap L^p_+ = \{0\}.$$

NFL is in general stronger than NA. In the companion FTAP: Infinite-Dimensional Strategies notebook, the two conditions are genuinely distinct when the strategy space is infinite-dimensional. Here they coincide:

Property 1 (No Arbitrage Implies Closedness)**.** *In the one-period model with d risky assets and payoffs in L^p, NA implies A is closed in L^p, so NA ⇔ NFL.*

This is the key structural fact: the strategy space $\mathbb{R}^d$ is finite-dimensional, so bounded sequences of strategies have convergent subsequences. The proof is contained in Property 2 below.

The Fundamental Theorem

Property 2 (Fundamental Theorem of Asset Pricing)**.** *The following conditions are equivalent:*

(a) $A \cap L^p_+ = \{0\}$ *(no arbitrage),*
(b) $A \cap L^p_+ = \{0\}$ *and* $A = \bar{A}$ *(no arbitrage and* A *is closed),*
(c) $\bar{A} \cap L^p_+ = \{0\}$ *(no free lunch),*
(d) there exists $\tilde{P} \sim P$ *with* $d\tilde{P}/dP \in L^q$ *such that* $\mathsf{E}_{\tilde{P}}[\Delta S] = 0$.

Condition (d) says there is an *equivalent martingale measure* $\tilde{P}$ under which each risky asset has zero expected return, with Radon-Nikodym density in the dual space L^q. The implications (b) ⇒ (c) ⇒ (a) are trivial; the substance is (a) ⇒ (b) and (c) ⇒ (d).

Two special cases are worth noting. When $p = 1$, the density $d\tilde{P}/dP \in L^\infty$ is bounded, recovering the result of Kabanov and Stricker (2001). When $p = 2$, the density is square-integrable: $d\tilde{P}/dP \in L^2$.

Proof Tools

The proof uses two results.

Lemma 0.1 (Hahn-Banach Separation in L^p)**.** *Let* $C \subseteq L^p$ *be a non-empty closed convex cone containing* 0*, and let* $x \in L^p_+$ *with* $x \notin C$*. Then there exists* $z \in L^q$ *such that*

$$\mathsf{E}(z\,y) \leq 0 < \mathsf{E}(z\,x) \quad \textit{for all } y \in C.$$

This follows from the Hahn-Banach separation theorem in the locally convex space L^p, combined with the duality $(L^p)^* = L^q$. When $p = 1$, the dual $(L^1)^* = L^\infty$ gives $z \in L^\infty$ normalizable to $z \leq 1$ a.s. When $p = 2$, the Hilbert-space self-duality $(L^2)^* = L^2$ means the separator lies in L^2 itself.

Lemma 0.2 (Halmos-Savage Theorem). *Let $\{\mu_\alpha : \alpha \in I\}$ be a family of measures on $(\Omega, \mathcal{F})$, each absolutely continuous with respect to P. If for every measurable A with $P(A) > 0$ there exists some α with $\mu_\alpha(A) > 0$, then there exists a countable subfamily with the same null sets as the full family.*

Proof of Property 2

(a) ⇒ (b): NA implies A is closed. Suppose $y^{(n)} \in A$ with $y^{(n)} \to y$ in L^p. Each $y^{(n)} \leq \theta^{(n)} \cdot \Delta S$ a.s. for some $\theta^{(n)} \in \mathbb{R}^d$. We show $\theta^{(n)}$ is bounded.

Let $N = \ker(\theta \mapsto \theta \cdot \Delta S) \subseteq \mathbb{R}^d$ and write $\theta^{(n)} = \phi^{(n)} + \eta^{(n)}$ with $\phi^{(n)} \in N^\perp$ and $\eta^{(n)} \in N$. Since $\eta^{(n)} \cdot \Delta S = 0$ a.s., the payoff satisfies $\theta^{(n)} \cdot \Delta S = \phi^{(n)} \cdot \Delta S$, so it suffices to show $\phi^{(n)}$ is bounded in $N^\perp$.

Suppose for contradiction that $\|\phi^{(n)}\| \to \infty$ along some subsequence. Normalize: $\hat{\phi}^{(n)} = \phi^{(n)}/\|\phi^{(n)}\|$ lies on the unit sphere of $N^\perp$, which is compact. Pass to a subsequence with $\hat{\phi}^{(n)} \to \hat{\phi} \in N^\perp$, $\|\hat{\phi}\| = 1$. Since $\hat{\phi}^{(n)} \to \hat{\phi}$ in $\mathbb{R}^d$ and $\Delta S \in (L^p)^d$, we have $\hat{\phi}^{(n)} \cdot \Delta S \to \hat{\phi} \cdot \Delta S$ in L^p. Since $y^{(n)}/\|\phi^{(n)}\| \to 0$ in L^p, passing to an a.s.-convergent subsequence of each gives

$$\hat{\phi} \cdot \Delta S \;\geq\; \lim_n \frac{y^{(n)}}{\|\phi^{(n)}\|} \;=\; 0 \quad \text{a.s.}$$

By NA, $\hat{\phi} \cdot \Delta S \in A \cap L^p_+ = \{0\}$, so $\hat{\phi} \in N$. But $\hat{\phi} \in N^\perp$ and $\|\hat{\phi}\| = 1$, giving $\hat{\phi} \in N \cap N^\perp = \{0\}$, a contradiction.

Hence $\phi^{(n)}$ is bounded in $N^\perp$, so $\theta^{(n)}$ is effectively bounded. Pass to a subsequence with $\phi^{(n)} \to \phi$. Then $\phi^{(n)} \cdot \Delta S \to \phi \cdot \Delta S$ in L^p, so passing to an a.s.-convergent subsequence, $y \leq \phi \cdot \Delta S$ a.s. and $y \in A$.

(b) ⇒ (c): Trivial since $A = \bar{A}$ under (b).

(c) ⇒ (d): Since (c) implies (a) and (a) implies (b), A is closed. The cone A is a closed convex cone in L^p satisfying $-L^p_+ \subseteq A$ and $A \cap L^p_+ = \{0\}$.

Step 1: Point-wise separation. Fix any $x \in L^p_+$ with $x \neq 0$; by NFL, $x \notin A$. By Lemma 0.1, there exists $z_x \in L^q$ such that

$$\mathsf{E}(z_x\, y) \leq 0 \quad \text{for all } y \in A, \qquad \mathsf{E}(z_x\, x) > 0.$$

To show $z_x \geq 0$ a.s.: since $-\mathbf{1}_{z_x<0} \in -L^p_+ \subseteq A$, we have $\mathsf{E}(z_x\, \mathbf{1}_{z_x<0}) \geq 0$. But $z_x < 0$ on $\{z_x < 0\}$, so $z_x\, \mathbf{1}_{z_x<0} \leq 0$ a.s., forcing $P(z_x < 0) = 0$. Normalize so that $\|z_x\|_q = 1$, and define $\mu_x(B) := \mathsf{E}(z_x\, \mathbf{1}_B)$ for $B \in \mathcal{F}$. In particular, for any B with $P(B) > 0$, taking $x = \mathbf{1}_B$ gives $\mu_{\mathbf{1}_B}(B) = \mathsf{E}(z_{\mathbf{1}_B}\, \mathbf{1}_B) > 0$, so the family $\{\mu_x : x \in L^p_+,\ x \neq 0\}$ charges every set of positive P-measure.

Step 2: Extract a countable equivalent subfamily. By Lemma 0.2, there exists a countable sequence $\{z_{x_n}\}_{n\geq 1}$ with the same null sets as the full family $\{z_x\}$.

Step 3: Construct the density. Define

$$\rho = \sum_{n=1}^{\infty} 2^{-n}\, z_{x_n}.$$

Since $\|z_{x_n}\|_q = 1$, the triangle inequality gives $\|\rho\|_q \leq \sum_{n=1}^{\infty} 2^{-n} = 1$, so $\rho \in L^q$. Moreover $\rho > 0$ a.s.: if $P(\rho = 0) > 0$, let $B = \{\rho = 0\}$; since each $z_{x_n} \geq 0$ and $\rho = \sum 2^{-n} z_{x_n}$, we get $z_{x_n} = 0$ a.s. on B for all n, so $\mu_{x_n}(B) = 0$ for all n. But $P(B) > 0$, so the family $\{\mu_x\}$ charges B (Step 1), contradicting the null-set equivalence of Step 2. Since $\rho > 0$ a.s. we have $\mathsf{E}(\rho) > 0$; set $d\tilde{P}/dP = \rho/\,\mathsf{E}(\rho) \in L^q$. Then $\tilde{P} \sim P$.

Step 4: $\tilde{P}$ is a martingale measure. Each z_{x_n} satisfies $\mathsf{E}(z_{x_n}\, y) \leq 0$ for all $y \in A$. Since $\pm\theta \cdot \Delta S \in R \subseteq A$ for every $\theta \in \mathbb{R}^d$, applying this to both signs gives $\mathsf{E}(z_{x_n}\, \theta \cdot \Delta S) = 0$ for all θ. By Hölder's inequality, $|\,\mathsf{E}(z_{x_n}\, \theta \cdot \Delta S)| \leq \|z_{x_n}\|_q \|\theta \cdot \Delta S\|_p = \|\theta \cdot \Delta S\|_p$, so the series $\sum 2^{-n}\, \mathsf{E}(z_{x_n}\, \theta \cdot \Delta S)$ converges absolutely and by linearity of expectation $\mathsf{E}(\rho\, \theta \cdot \Delta S) = 0$ for all θ, hence $\mathsf{E}_{\tilde{P}}[\Delta S] = 0$.

(d) ⇒ (a): Let $\xi \in A \cap L^p_+$, so $0 \leq \xi \leq \theta \cdot \Delta S$ a.s. for some θ. Then $0 \leq \mathsf{E}_{\tilde{P}}[\xi] \leq \mathsf{E}_{\tilde{P}}[\theta \cdot \Delta S] = \theta \cdot \mathsf{E}_{\tilde{P}}[\Delta S] = 0$. Since $\xi \geq 0$ and $\tilde{P} \sim P$, we get $\xi = 0$ a.s. □

Why Infinite Strategies Are Harder

The simplicity of this proof rests on one structural feature: the strategy space $\mathbb{R}^d$ is finite-dimensional.

NA implies closedness via compactness. The closedness argument works by contradiction. If the norms of effective strategies (those in $N^\perp$) grow without bound, normalizing them produces a sequence on the unit sphere of $N^\perp$. That sphere is **compact,** so Bolzano-Weierstrass supplies a convergent subsequence. The limit direction then yields a nonnegative payoff, an arbitrage, contradicting NA. Compactness is what converts NA into closedness of A.

In the companion notebook, the strategy space is an infinite-dimensional subspace $X \subseteq L^p$: **bounded sets in infinite-dimensional Banach spaces are not compact,** Bolzano-Weierstrass fails, and one can exhibit markets where NA holds but A is not closed. This is why NFL must be imposed as an explicit hypothesis rather than derived from NA.

The same issue appears in **dynamic and continuous-time** models. Dybvig and Huang (1988) show that a **nonnegative wealth constraint** ($W_t \geq 0$ a.s. for all t) rules out all arbitrage — including doubling strategies — precisely because it ensures closedness of the attainable payoff cone. Like finite-dimensionality in the static model, it is a structural condition that does the work NFL must do explicitly when the strategy space is unconstrained.

References

Dybvig, Philip H., and Chi-fu Huang. 1988. "Nonnegative Wealth, Absence of Arbitrage, and Feasible Consumption Plans." *Review of Financial Studies* 1 (4): 377–401.

Kabanov, Yuri M., and Christophe Stricker. 2001. "A Teachers' Note on No-Arbitrage Criteria." In *Séminaire de Probabilités XXXV*, vol. 1755. Lecture Notes in Mathematics. Springer.