Consumption Based Asset Pricing

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Introduction

In a previous notebook we introduced the **stochastic discount factor** (SDF) as the object that prices any payoff through

$$p_t = \mathrm{E}_t(m_{t+1} x_{t+1}).$$

This notebook asks where that SDF comes from in equilibrium. The consumption-based asset pricing model gives one answer: when investors choose consumption optimally over time and across states, the SDF is proportional to the intertemporal marginal rate of substitution.

This is not the only way to generate discount factors in equilibrium. For example, in The Fisher Model we study a simple general-equilibrium environment in which production and consumption decisions jointly determine discount factors. The present notebook isolates the consumption side of that logic and shows how preferences alone deliver a pricing kernel.

Our goal is therefore not to reintroduce the pricing equation, but to derive an economically meaningful specification for m_{t+1} from preferences over consumption. Once that link is in place, asset prices and expected returns can be interpreted in terms of how payoffs covary with marginal utility.

This framework is central to modern asset pricing because it connects equilibrium behavior, risk premia, and the valuation of cash flows through a single object: marginal utility growth. In this sense, the consumption-based model provides one of the canonical economic foundations for the SDF approach developed in Lucas (1978) and Breeden (1979).

As before, we study the price p_t at time t of a payoff x_{t+1} paid at time $t+1$. The payoff will in general be random, and therefore unknown at time t.

For example, if you purchase a stock at time t your payoff at time $t+1$ will be the price p_{t+1} at which you can sell the stock plus possibly a dividend d_{t+1}, or $x_{t+1} = p_{t+1} + d_{t+1}$.

Example 1. Consider a stock that currently trades for 7. The table below shows the dividend and price expected next period for different scenarios.

	Probability	Dividend	Price	Payoff
Boom	0.3	1.0	10	11.0
Normal	0.5	0.8	7	7.8
Recession	0.2	0.7	4	4.7

The payoff of purchasing the stock today is therefore a random variable defined in some finite probability space (Ω, P). Note that the expected payoff is 8.14, which implies an expected return of 16.29%.

The big question in finance is then how to determine the SDF, and therefore the price p_t of the asset at time t. In Example 1, the payoffs of the stock are generic and could have been the payoffs of many other assets. Thus, the asset pricing theory we develop here applies not only to stocks, but also to bonds and derivatives.

This treatment follows closely Chapter 1 in Cochrane (2009).

Pricing Assets

We now derive the stochastic discount factor implied by optimal consumption choice. Start from the investor's intertemporal utility and ask how utility changes if she perturbs an otherwise optimal portfolio by buying a small additional amount of one asset.

A simple way to model the tradeoff between consuming today versus tomorrow is to write

$$U(c_t, c_{t+1}) = u(c_t) + \delta \, \mathrm{E}_t[u(c_{t+1})], \tag{1}$$

where $\mathsf{E}_t[\cdot]$ denotes the expectation conditional on the information available at time t, $u(\cdot)$ is an increasing and concave function of consumption, and $\delta < 1$ is a discounting factor.

In the expression above, $U(c_t, c_{t+1})$ is the utility of consuming c_t today and c_{t+1} tomorrow. The investor must decide how much to consume today, how much to save, and how to allocate savings across assets. We now take an optimal consumption-investment plan as given and ask what restriction optimality imposes on the price of any asset.

Consider now the case in which the investor deviates from her optimal investment rule and decides to purchase ξ additional shares of an asset that currently trades for p_t. Her consumption at time t drops to $c_t - \xi p_t$ since she needs the money to purchase the additional shares, whereas her consumption at time $t+1$ changes to $c_{t+1} + \xi x_{t+1}$ since she will receive an additional payoff from the asset she just purchased.

Given the original levels of consumption c_t and c_{t+1}, the resulting utility can be written as a function of ξ,

$$U(\xi) = u(c_t - \xi p_t) + \delta\, \mathsf{E}_t[u(c_{t+1} + \xi x_{t+1})].$$

If the original allocation is optimal, then $\xi = 0$ must maximize this function. The first-order condition for ξ is

$$U'(\xi) = -p_t u'(c_t - \xi p_t) + \delta\, \mathsf{E}_t[x_{t+1} u'(c_{t+1} + \xi x_{t+1})] = 0.$$

Evaluating the first-order condition at $\xi = 0$ gives

$$-p_t u'(c_t) + \delta\, \mathsf{E}_t[x_{t+1} u'(c_{t+1})] = 0,$$

or

$$p_t = \mathsf{E}_t\left[\delta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right]. \tag{2}$$

Equation (2) identifies the SDF in this model as the intertemporal marginal rate of substitution. The resulting pricing formula, $p = \mathsf{E}(mx)$, should now look familiar from the previous notebook, but here it acquires an explicit economic interpretation. An asset commands a high price if it pays well in states where marginal utility is high, that is, when consumption is low. By contrast, an asset that pays mainly in good times provides little insurance and therefore carries a lower price.

The term

$$m_{t+1} = \delta \frac{u'(c_{t+1})}{u'(c_t)}$$

is the **stochastic discount factor** (SDF) or **pricing kernel** implied by optimal consumption choice.

Property 1 (Consumption-Based Pricing Kernel). *In the consumption-based model, the price of an asset that pays x_{t+1} at time $t+1$ is*

$$p_t = \mathrm{E}_t(m_{t+1} x_{t+1}), \tag{3}$$

where

$$m_{t+1} = \delta \frac{u'(c_{t+1})}{u'(c_t)}. \tag{4}$$

This is the familiar SDF pricing formula, now specialized to an equilibrium in which the pricing kernel is pinned down by consumption growth. We will often suppress time subscripts and write

$$p = \mathrm{E}(mx), \tag{5}$$

where it is understood that p denotes the price today of a payoff x paid next period.

If x is a random variable defined on a finite probability space (Ω, P), the pricing equation can be written as

$$p = \sum_{w \in \Omega} \mathrm{P}(\omega) m(\omega) x(\omega), \tag{6}$$

where ω denotes a state of the world, $\mathrm{P}(\omega)$ is the probability of the outcome ω occurring, and $x(\omega)$ is the payoff if ω happens.

Example 2. Consider an investor with power utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

In that case, the SDF becomes

$$m_{t+1} = \delta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}.$$

The investor's current consumption is $c_t = 6.5$, and is considering investing in two assets X and Y. The table below presents the probabilities of different scenarios, along with the future consumption and payoffs of the assets.

	Probability	Consumption	Payoff X	Payoff Y
Boom	0.3	9.0	9.8	6.0
Normal	0.5	6.7	8.3	5.0
Recession	0.2	5.4	6.5	7.1

If $\gamma = 4$ and $\delta = 0.95$, we can compute the stochastic discount factor (SDF) for each scenario as

	Probability	SDF
Boom	0.3	0.258
Normal	0.5	0.842
Recession	0.2	1.994

Using equation (6), we obtain $p(x) = 6.85$ and $p(y) = 5.4$. Expected returns are the expected payoffs divided by price minus one, so $\mathrm{E}(r^x) = 22.57\%$ and $\mathrm{E}(r^y) = 5.91\%$.

So far we have measured prices and payoffs in units of real consumption. The stochastic discount factor in (4) is therefore a real discount factor.

The same logic can be translated into nominal units once we adjust for inflation.

Let $p_t = p_t^*/\Pi_t$ and $x_{t+1} = x_{t+1}^*/\Pi_{t+1}$ where Π_t denotes the price level and we use asterisks to denote nominal quantities. Then,

$$\frac{p_t^*}{\Pi_t} = \mathrm{E}_t \left(m_{t+1} \frac{x_{t+1}^*}{\Pi_{t+1}} \right).$$

The previous expression implies that

$$p_t^* = \mathsf{E}_t\left(m_{t+1}^* x_{t+1}^*\right),$$

where $m_{t+1}^* = m_{t+1}\frac{\Pi_t}{\Pi_{t+1}}$. We obtain the same equation as before but now expressed in nominal terms. In the analysis, m_{t+1}^* plays the role of a nominal discount factor.

Prices and Returns

In the previous notebook we saw how the SDF pins down returns. Here we briefly recall the key identities and apply them to the consumption-based SDF derived above.

We will use

$$R_{t+1} = \frac{x_{t+1}}{p_t}$$

to denote the gross return on an asset. For example, if an investment of \$100 generates \$105, the gross return is $R = 1.05$, whereas the net return is $r = 5\%$.

Substituting $R_{t+1} = x_{t+1}/p_t$ into the pricing equation gives

$$1 = \mathsf{E}(mR). \tag{7}$$

Thus, for any asset i,

$$\mathsf{E}(mR^i) = 1. \tag{8}$$

A risk-free asset pays next period $x = 1$ in every state, so its return R^f is constant. Therefore

$$R^f = \frac{1}{\mathsf{E}(m)}.$$

In the consumption-based model, the risk-free rate is therefore determined by the mean of the marginal-rate-of-substitution process.

Example 3. Using the data of Example 2, we find that $\mathsf{E}(m) = 0.8972$. Therefore,

$$R^f = \frac{1}{0.8972} = 1.1146,$$

or a net return of 11.46% per period.

Compensation for Time and Risk

The example also helps clarify how the model separates the time value of money from risk adjustment. Starting from the pricing equation,

$$\begin{aligned} p &= \mathsf{E}(mx) = \mathsf{E}(m)\,\mathsf{E}(x) + \mathsf{Cov}(m, x) \\ &= \frac{\mathsf{E}(x)}{R^f} + \mathsf{Cov}(m, x). \end{aligned} \tag{9}$$

The first term discounts expected cash flows at the risk-free rate. The second term is the risk adjustment. Prices are therefore high when the risk-free rate is low and when payoffs covary positively with the SDF, that is, when they are valuable in bad times.

In return space, the same idea becomes

$$1 = \mathsf{E}(mR^i) = \mathsf{E}(m)\,\mathsf{E}(R^i) + \mathsf{Cov}(m, R^i).$$

Rearranging yields

$$\mathsf{E}(R^i) - R^f = -\frac{\mathsf{Cov}(R^i, m)}{\mathsf{E}(m)} = \beta_{i,m}\lambda_m, \tag{10}$$

where $\beta_{i,m} = \dfrac{\mathsf{Cov}(R^i, m)}{\mathsf{V}(m)}$ and $\lambda_m = -\dfrac{\mathsf{V}(m)}{\mathsf{E}(m)} < 0.$

Equation (10) is a beta pricing relation in which the systematic factor is the SDF itself. Assets with high $\beta_{i,m}$ pay off when marginal utility is high, so they hedge bad times and require lower expected returns.

Since $m_{t+1} = \delta u'(c_{t+1})/u'(c_t)$, this is equivalent to a beta pricing model in which risk is measured by covariance with future marginal utility.

Multi-Period Asset Pricing

Consider now a stream of consumption $\{c_{t+j}\}_{j=0}^{\infty}$. We can extend the utility function defined earlier as

$$V_t = \mathsf{E}_t \sum_{j=0}^{\infty} \delta^j u(c_{t+j}). \tag{11}$$

Note that we can also write (11) as

$$\begin{aligned} V_t &= u(c_t) + \delta \, \mathsf{E}_t \sum_{j=0}^{\infty} \delta^j u(c_{t+1+j}) \\ &= u(c_t) + \delta \, \mathsf{E}_t \, \mathsf{E}_{t+1} \sum_{j=0}^{\infty} \delta^j u(c_{t+1+j}) \\ &= u(c_t) + \delta \, \mathsf{E}_t \, V_{t+1}. \end{aligned} \tag{12}$$

Equation (12) is the corresponding Bellman equation for equation (11).

Now consider a stream of dividends $\{D_{t+j}\}_{j=1}^{\infty}$. The same perturbation analysis can be used to show that

$$p_t = \mathsf{E}_t \sum_{j=1}^{\infty} \delta^j \frac{u'(c_{t+j})}{u'(c_t)} D_{t+j}. \tag{13}$$

Therefore, the stochastic discount factor for a payoff received at time $t+j$ is

$$m_{t+j} = \delta^j \frac{u'(c_{t+j})}{u'(c_t)}.$$

We can use (13) to price any asset with cash flows paid at time $T > 1$. For example, consider a real discount bond expiring with time-to-maturity time T and face value 1 unit of consumption. The price of the bond is given by

$$B_t = \mathsf{E}_t(m_{t+T}).$$

Similarly, the price of a nominal discount bond expiring with time-to-maturity time T and face value \$1 is

$$B_t^* = \mathsf{E}_t\left(m_{t+T} \frac{\Pi_t}{\Pi_{t+T}}\right) = \mathsf{E}_t(m_{t+T}^*).$$

Finally, the price of a call option with strike K and maturity T is given by

$$c_t = \mathrm{E}_t(m^*_{t+T}(S_{t+T} - K)^+).$$

The same consumption-based pricing kernel therefore extends naturally from one-period claims to bonds, equities with multiple cash flows, and derivatives.

References

Breeden, Douglas T. 1979. "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities." *Journal of Financial Economics* 7 (3): 265–96.

Cochrane, John. 2009. *Asset Pricing: Revised Edition*. Princeton university press.

Lucas, Robert E. 1978. "Asset Prices in an Exchange Economy." *Econometrica* 46 (6): 1429–45.