

No-Arbitrage and Positive SDFs

The [previous notebook](#) derived the stochastic discount factor from the law of one price alone. That gave a unique SDF x^* in the space of traded payoffs, the risk-free rate, risk premia, the Hansen-Jagannathan bound, and the link between the SDF and the mean-variance frontier, but no guarantee that any SDF is positive. This notebook adds the **principle of no-arbitrage** and shows that it is exactly the condition for a strictly positive SDF to exist, with foundations in work such as Kreps (1981). Positivity is what lets an SDF price payoffs that are not traded, such as options, without creating arbitrage.

The Principle of No-Arbitrage

The law of one price disciplines prices across portfolios: equal payoffs must have equal prices. But LOOP says nothing about the relationship between prices and the *sign* of payoffs. A pricing functional can be perfectly linear, satisfying LOOP, and yet assign a non-positive price to a payoff that is non-negative in every state. An investor facing such an opportunity would want to hold an unbounded position: buying costs nothing (or generates an immediate cash inflow) while the payoff can only be zero or positive. In competitive markets, unbounded demand is inconsistent with equilibrium clearing, so such opportunities cannot persist.

Assumption 1 (Principle of No-Arbitrage). *The price of a payoff that is non-negative in all states and strictly positive in at least one state of the world must be positive.*

To see that PNA is strictly stronger than LOOP, consider a market with a single risky asset x_1 that pays \$1 in state 1 and \$0 in state 2. Any pricing functional of the form $p(ax_1) = a\pi_1$ is linear and therefore satisfies LOOP regardless of the sign of π_1 . But if $\pi_1 \leq 0$, buying x_1 costs nothing (or generates income) while delivering a non-negative payoff. LOOP is silent on this because there is no inconsistency across portfolios; the problem is that the price assigns the wrong sign to a desirable payoff.

More generally, PNA adds the requirement that prices respect *dominance*: a payoff that weakly improves on zero in every state, and strictly improves in at least one, must carry a strictly positive price. This is the minimal condition consistent with the behavior of an investor who strictly prefers more consumption to less in every state of the world.

The principle of no-arbitrage (PNA) is a strictly stronger assumption than LOOP, as the next property shows.

Property 1.

$$PNA \Rightarrow LOOP.$$

Proof

We will prove this claim by contradiction. Assume that PNA holds but not LOOP. A violation of LOOP implies that the price of a zero payoff is not zero: if $p(x) \neq \sum_i a_i \pi_i$ for some $x = \sum_i a_i x_i \in X$, then the zero payoff $x - \sum_i a_i x_i = 0$ is priced at $p(x) - \sum_i a_i \pi_i \neq 0$. Without loss of generality, assume $p_0 = p(x) - \sum_i a_i \pi_i > 0$. Take any payoff $x^+ \in X$ that is non-negative and non-zero, with price $p > 0$. Form a portfolio that buys one unit of x^+ and sells $n > \frac{p}{p_0}$ units of this zero-payoff portfolio. The cost of that portfolio is

$$\pi = p - np_0 < p - \frac{p}{p_0} p_0 = 0,$$

but its payoff is identical to x^+ , which is non-negative and non-zero, violating PNA.

This contradiction establishes the result. □

The reverse implication does not hold: a market can satisfy LOOP while still admitting free lunches, as the single-asset example above illustrates. PNA therefore imposes a genuinely additional restriction on the pricing functional beyond linearity. In particular, it rules out not only mispricing across equivalent portfolios, but also prices that attach non-positive value to payoffs that are everywhere non-negative. This distinction is important because return-based formulas implicitly assume that the asset price is strictly positive. If a non-negative payoff had zero or negative price, its gross return would be undefined or economically ill-behaved. That is why arbitrage restrictions are stated first in payoff-price language, and only afterwards translated into statements about returns.

A more important consequence of PNA is that it is equivalent to the existence of a strictly positive SDF. Under LOOP, an SDF m prices every traded payoff via $p(x) = E(mx)$, but a generic SDF need not be positive in every state. If some component satisfies $m(s) \leq 0$, then the state- s payoff receives non-positive marginal value. In complete markets, the [Arrow-Debreu security](#) e_s would then have price $p(e_s) = E(me_s) = q(s)m(s) \leq 0$, immediately violating PNA. More generally, the positive-SDF theorem below shows that ruling out such statewise negative valuations on the traded payoff space is exactly equivalent to PNA.

Property 2 (No-Arbitrage and Strictly Positive SDFs).

$$PNA \Leftrightarrow \exists \text{ a valid SDF } m > 0.$$

A proof is provided in [the appendix](#). Even in a finite state space, this is not an elementary result: the argument already requires a geometric reformulation of the pricing problem and an application of the Separating Hyperplane Theorem. That depth is part of the point. The equivalence says that no-arbitrage is exactly the condition that the market assigns a positive shadow value to wealth in every state: no state is “throwaway.” In complete markets ($N = S$), PNA pins down a *unique* strictly positive SDF, since $X = L$ leaves no room for the orthogonal residual ε in the decomposition $m = x^* + \varepsilon$ of the [previous notebook](#). PNA also guarantees that the price of a traded risk-free asset is positive, as the previous notebook assumed.

Positive SDFs and Option Prices

No-arbitrage requires that *some* SDF be strictly positive. It does not require x^* , the SDF in the space of traded payoffs, to be positive, and the difference matters as soon as we price a payoff that is not traded.

Example 1. Suppose there are three equally likely states, a bond with payoff $x_1 = [1, 1, 1]'$ and a stock with payoff $x_2 = [0, 1, 2]'$. Take the SDF $m = [2, 0.2, 0.2]'$, which is strictly positive, so the market has no arbitrage. Prices are $\pi_1 = E(mx_1) = 0.8$ and $\pi_2 = E(mx_2) = 0.2$. The second-moment matrix is

$$E(\mathbf{x}\mathbf{x}') = \begin{pmatrix} 1 & 1 \\ 1 & \frac{5}{3} \end{pmatrix},$$

so the projection of m onto the traded payoffs is

$$x^* = \pi' E(\mathbf{x}\mathbf{x}')^{-1} \mathbf{x} = 1.7 x_1 - 0.9 x_2 = [1.7, 0.8, -0.1]'$$

It is negative in the third state even though the market has no arbitrage. Now consider a call option on the stock with strike 1, which pays $c = \max(x_2 - 1, 0) = [0, 0, 1]'$ and is not traded. The SDF x^* would price it at $E(x^* c) = -\frac{1}{30}$, while m prices it at $E(mc) = \frac{1}{15}$. Both SDFs price the bond and the stock correctly, but only the positive one extends to the option without creating an arbitrage: buying the call at a negative price delivers a nonnegative payoff for a cash inflow today. $\square$

The SDF x^* is proportional to the minimum-second-moment return R^* , which lies on the mean-variance frontier, so pricing the option with x^* amounts to pricing it by beta with respect to a frontier return. Dybvig and Ingersoll (1982) prove the general version of this example: if mean-variance pricing holds for all assets in a complete market and the market return can be high enough, a call option on the market with a high strike has a negative price, an arbitrage. Beta pricing, which rests only on the law of one price, can therefore be extended to options only when the SDF it implies is strictly positive.

Appendix

Proof of Property 2

The proof relies on the following geometric result.

Lemma 1 (Separating Hyperplane Theorem). *If A and B are non-empty disjoint convex subsets of $\mathbb{R}^n$, there exists a non-zero vector $\phi \in \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$ such that*

$$\phi \cdot a \leq \alpha \leq \phi \cdot b \quad \text{for all } a \in A, b \in B.$$

See, for example, Boyd and Vandenberghe (2004).

Road map. The direction ($\Leftarrow$) is direct: if $m > 0$, every non-negative non-zero payoff receives a positive price, ruling out arbitrage. For ($\Rightarrow$), we argue by contradiction. We reformulate the problem as finding a strictly positive state-price vector that correctly prices all assets. Assume no such vector exists, so the plane of valid state prices and the positive orthant are disjoint. The Separating Hyperplane Theorem then places a hyperplane between them, producing a separating normal vector. We show this normal is the payoff of some traded portfolio, and that the three properties forced by the separation (non-negative payoff, non-zero payoff, and non-positive cost) make that portfolio an arbitrage, contradicting PNA.

($\Leftarrow$): Suppose $m > 0$ and let $x \in X$ satisfy $x \geq 0$ and $x \neq 0$. Since x is non-negative but not identically zero, there is at least one state s^* with $x(s^*) > 0$. In the sum

$$p(x) = E(mx) = \sum_{s=1}^S q(s) m(s) x(s),$$

every term is non-negative (because $q(s) > 0$, $m(s) > 0$, and $x(s) \geq 0$), and the term at $s = s^*$ is strictly positive. Hence $p(x) > 0$, which is exactly PNA.

($\Rightarrow$): Suppose PNA holds. Since $\text{PNA} \Rightarrow \text{LOOP}$ (by Property 1), the SDF x^* of the [previous notebook](#) exists. We must show that some SDF is strictly positive in every state.

Reformulation in terms of state prices. Recall that the state-price vector $\psi \in \mathbb{R}^S$ is given by

$$\psi(s) = q(s) m(s).$$

Since $q(s) > 0$, we have $m(s) > 0$ if and only if $\psi(s) > 0$, so proving $m > 0$ is equivalent to finding $\psi \gg 0$. In terms of ψ , the pricing condition $\pi_i = E(mx_i)$ becomes a Euclidean dot product $\pi_i = \psi \cdot x_i$. Stacking the N conditions into the $S \times N$ payoff matrix D (whose i -th column is x_i),

$$D = \begin{bmatrix} x_1(1) & \cdots & x_N(1) \\ \vdots & \ddots & \vdots \\ x_1(S) & \cdots & x_N(S) \end{bmatrix},$$

the pricing constraints become $D'\psi = \pi$. Note that the traded payoff space is $X = \text{col}(D) = \{D\theta : \theta \in \mathbb{R}^N\}$, so its orthogonal complement is $X^\perp = \{e \in \mathbb{R}^S : D'e = 0\}$. We therefore need to show that $D'\psi = \pi$ has a solution with $\psi \gg 0$. Equivalently, if $\mathbf{x}$ denotes the column

vector of traded payoff random variables introduced earlier, then the s -th row of D is $\mathbf{x}(s)'$, so D is the state-by-state realization matrix of $\mathbf{x}$.

Geometric setup. Define

$$\Psi = \{\psi \in \mathbb{R}^S : D' \psi = \pi\},$$

the flat plane of all state-price vectors consistent with observed prices. It is non-empty because $\psi^*(s) = q(s) x^*(s) \in \Psi$. The open positive orthant $\mathbb{R}_{++}^S$ is the set of strictly positive state prices. Our goal is to show that $\Psi \cap \mathbb{R}_{++}^S \neq \emptyset$ under PNA.

Contradiction. Suppose $\Psi \cap \mathbb{R}_{++}^S = \emptyset$. Since both sets are non-empty and convex, the **Separating Hyperplane Theorem** yields a non-zero $\phi \in \mathbb{R}^S$ and $\alpha \in \mathbb{R}$ with

$$\phi \cdot \psi \leq \alpha \leq \phi \cdot y \quad \text{for all } \psi \in \Psi, y \in \mathbb{R}_{++}^S.$$

Step 1: $\phi \geq 0$ and $\alpha \leq 0$. The positive orthant is a cone: if $y \in \mathbb{R}_{++}^S$, then $ty \in \mathbb{R}_{++}^S$ for every $t > 0$. Hence the separation inequality gives

$$\alpha \leq \phi \cdot (ty) = t(\phi \cdot y) \quad \text{for all } t > 0.$$

Letting $t \rightarrow 0^+$ yields $\alpha \leq 0$. Next, suppose for contradiction that $\phi(s) < 0$ for some state s . For any $\varepsilon > 0$ and any $t > 0$, the vector

$$y = \varepsilon \mathbf{1} + t e_s$$

lies in $\mathbb{R}_{++}^S$, so separation implies

$$\alpha \leq \phi \cdot y = \varepsilon \phi \cdot \mathbf{1} + t \phi(s).$$

Since $\phi(s) < 0$, the right-hand side tends to $-\infty$ as $t \rightarrow \infty$, contradicting the fixed lower bound α . Therefore every coordinate satisfies $\phi(s) \geq 0$, i.e., $\phi \geq 0$.

Step 2: $\phi \in X$, so ϕ is the payoff of some portfolio. Fix any $\psi_0 \in \Psi$. Since

$$\Psi = \psi_0 + X^\perp,$$

every vector of the form $\psi_0 + te$ with $e \in X^\perp$ and $t \in \mathbb{R}$ also lies in Ψ . Therefore

$$\phi \cdot (\psi_0 + tv) = \phi \cdot \psi_0 + t(\phi \cdot v) \leq \alpha \quad \text{for all } v \in X^\perp, t \in \mathbb{R}.$$

A linear function of t that is bounded above for all $t \in \mathbb{R}$ must have zero slope, so $\phi \cdot v = 0$ for every $v \in X^\perp$. Hence $\phi \in (X^\perp)^\perp = X$, and therefore there exists a portfolio $\theta \in \mathbb{R}^N$ with $\phi = D\theta$. For any $\psi \in \Psi$,

$$\phi \cdot \psi = (D\theta) \cdot \psi = \theta \cdot (D'\psi) = \theta \cdot \pi = \pi \cdot \theta,$$

so $\phi \cdot \psi$ is constant on Ψ . Since separation gives $\phi \cdot \psi \leq \alpha$ for every $\psi \in \Psi$, it follows that

$$\pi \cdot \theta \leq \alpha.$$

Step 3: Arbitrage. The portfolio θ satisfies:

- *Non-negative payoff in every state:* $D\theta = \phi \geq 0$.
- *Strictly positive payoff in at least one state:* $\phi \neq 0$ (since ϕ is a non-zero separating vector).
- *Non-positive cost:* $\pi \cdot \theta \leq \alpha \leq 0$.

This is precisely an arbitrage, contradicting PNA.

Conclusion. The assumption $\Psi \cap \mathbb{R}_{++}^S = \emptyset$ leads to an arbitrage, so it must be false. Hence there exists $\psi \gg 0$ with $D'\psi = \pi$, and the SDF $m(s) = \psi(s)/q(s) > 0$ is strictly positive in every state. □

References

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