

Beta Pricing with Frontier Portfolios

Introduction

In the [previous notebook](#) we characterized the minimum-variance frontier (MVF) for an economy with n risky assets. We found that the set of frontier portfolios forms a hyperbola in the (σ, μ) space and that any two frontier portfolios span the entire frontier. In this notebook we exploit the spanning property to establish a fundamental pricing result: the expected return of any asset is determined solely by how much it co-moves with any given frontier portfolio.

The key insight is that every asset can be decomposed into a frontier component and an idiosyncratic residual. The residual, by construction, has zero covariance with all frontier portfolios and therefore carries no compensation in expected returns — it is unpriced risk. What matters for pricing is only the systematic exposure to the frontier.

We develop this idea in three steps. First, we define the idiosyncratic risk of an asset and show it is uncorrelated with every frontier portfolio. Second, we establish that for every frontier portfolio p (other than the minimum variance portfolio) there exists a unique zero-covariance frontier portfolio z . Third, we combine these results to derive the beta-pricing formula

$$E(r_i) = \mu_z + \beta_i(E(r_p) - \mu_z),$$

where $\beta_i = \text{Cov}(r_i, r_p) / \sigma^2(r_p)$ and μ_z is the expected return of the zero-covariance portfolio of p . The [next notebook](#) shows how this formula simplifies when a risk-free asset is present.¹

¹Beta pricing needs only the law of one price, which makes portfolio returns weighted averages of asset returns ([Definition 1](#) of the previous notebook). It does not need the absence of arbitrage, and it does not imply it. Beta pricing with respect to p says that a stochastic discount factor linear in r_p , $m = a - b r_p$, prices every asset. When r_p is unbounded above, m is negative in some states. That is harmless for the n assets, but extending m to payoffs outside their span can assign a negative price to a nonnegative payoff. Dybvig and Ingersoll (1982) show that if mean-variance pricing holds for all assets in a complete market and the market return can be high enough, a call option on the market with a high strike has a negative price, an arbitrage. Beta pricing can therefore price options without creating arbitrage only when its stochastic discount factor is strictly positive.

Beta Pricing

Property 1 (Idiosyncratic Risk). *Consider an asset whose expected return is μ_i . We define the idiosyncratic risk of the asset as the difference between its return, and the return of a frontier portfolio that has the same expected return. It turns out that the idiosyncratic risk of any asset is uncorrelated with all frontier portfolios.*

Consider an arbitrary frontier portfolio with expected return μ_p and pick any asset or portfolio with expected return μ_i . The covariance between r_p and r_i is

$$\begin{aligned}\text{Cov}(r_i, r_p) &= \mathbf{w}_i' \mathbf{V} \mathbf{w}_p = \mathbf{w}_i' \mathbf{V} (\mathbf{a} + \mu_p \mathbf{b}) = \mathbf{w}_i' \left[\frac{1}{D} (B\mathbf{1} - A\mathbf{e}) + \frac{\mu_p}{D} (C\mathbf{e} - A\mathbf{1}) \right] \\ &= \frac{1}{D} (B - A\mu_p - A\mu_i + C\mu_i\mu_p).\end{aligned}$$

This shows that two assets with the *same expected return* will have the same covariance with a given frontier portfolio p . This seemingly marginal property is at the heart of modern asset pricing. Given a frontier portfolio, what determines the *expected return* of any asset is how much it covaries with the frontier portfolio regardless of its total risk.

To look at this result in more detail, consider the frontier portfolio $r_{p,i}$ with the same expected return as asset i . Define the residual

$$\varepsilon_i = r_i - r_{p,i}. \quad (1)$$

By construction, the residual has zero mean

$$E(\varepsilon_i) = E(r_i) - E(r_{p,i}) = 0,$$

and if we pick any frontier portfolio p we also have that

$$\text{Cov}(r_p, \varepsilon_i) = \text{Cov}(r_p, r_i) - \text{Cov}(r_p, r_{p,i}) = 0.$$

That is, the residual is unrelated to the returns of frontier portfolios. Since we just saw that the expected return of an asset depends only on how it covaries with a frontier portfolio, the residual is not priced, hence the name idiosyncratic risk.

Property 2 (Zero-Covariance). *For a given frontier portfolio p with expected return μ_p and variance σ_p^2 , we can always find (except for the minimum variance portfolio) a unique frontier portfolio z with expected return μ_z that is uncorrelated with p , i.e.*

$$\text{Cov}(r_z, r_p) = 0.$$

Just draw a line that is tangent to the minimum-variance frontier at the point (σ_p, μ_p) . The intercept of this line with the vertical axis gives μ_z .

Using the variance expression $\sigma^2 = (B - 2A\mu + C\mu^2)/D$, we can compute the derivative of σ with respect to μ ,

$$\frac{d\sigma}{d\mu} = \frac{d\sqrt{\sigma^2}}{d\mu} = \frac{1}{2\sigma} \frac{d\sigma^2}{d\mu} = \frac{C\mu - A}{D\sigma}. \quad (2)$$

Denote by m_p the slope coefficient of the minimum variance frontier at the point (σ_p, μ_p) . Equation (2) implies that

$$\frac{\sigma_p}{m_p} = \frac{C\mu_p - A}{D},$$

since $d\sigma/d\mu = 1/(d\mu/d\sigma) = 1/m$.

The equation of the tangent line to the minimum variance frontier at point (σ_p, μ_p) is given by

$$\mu - \mu_p = m_p(\sigma - \sigma_p).$$

Denote by μ_z the intercept of this line with the vertical axis. Then we have that

$$\mu_z = \mu_p - m_p\sigma_p.$$

We can now compute the covariance of r_p and a frontier portfolio r_z with expected return μ_z .

$$\begin{aligned} \text{Cov}(r_z, r_p) &= \frac{1}{D} (B - A\mu_p - A\mu_z + C\mu_z\mu_p) \\ &= \frac{1}{D} (B - 2A\mu_p + C\mu_p^2) - \frac{\mu_p - \mu_z}{D} (C\mu_p - A) \\ &= \sigma_p^2 - m_p\sigma_p \left(\frac{C\mu_p - A}{D} \right) = \sigma_p^2 - m_p\sigma_p \left(\frac{\sigma_p}{m_p} \right) = 0. \end{aligned}$$

Obviously, the covariance of the minimum variance portfolio with any other frontier portfolio is equal to its variance, so it is impossible to find a frontier portfolio that has zero-covariance with it.²

Figure 1 shows the construction for the frontier portfolio p with $\mu_p = 14\%$. The tangent at p meets the vertical axis at $\mu_z \approx 5.7\%$, and the zero-covariance portfolio z is the frontier portfolio with that mean. Because p is on the efficient branch, μ_z is below the mean of the MVP and z lies on the inefficient branch.

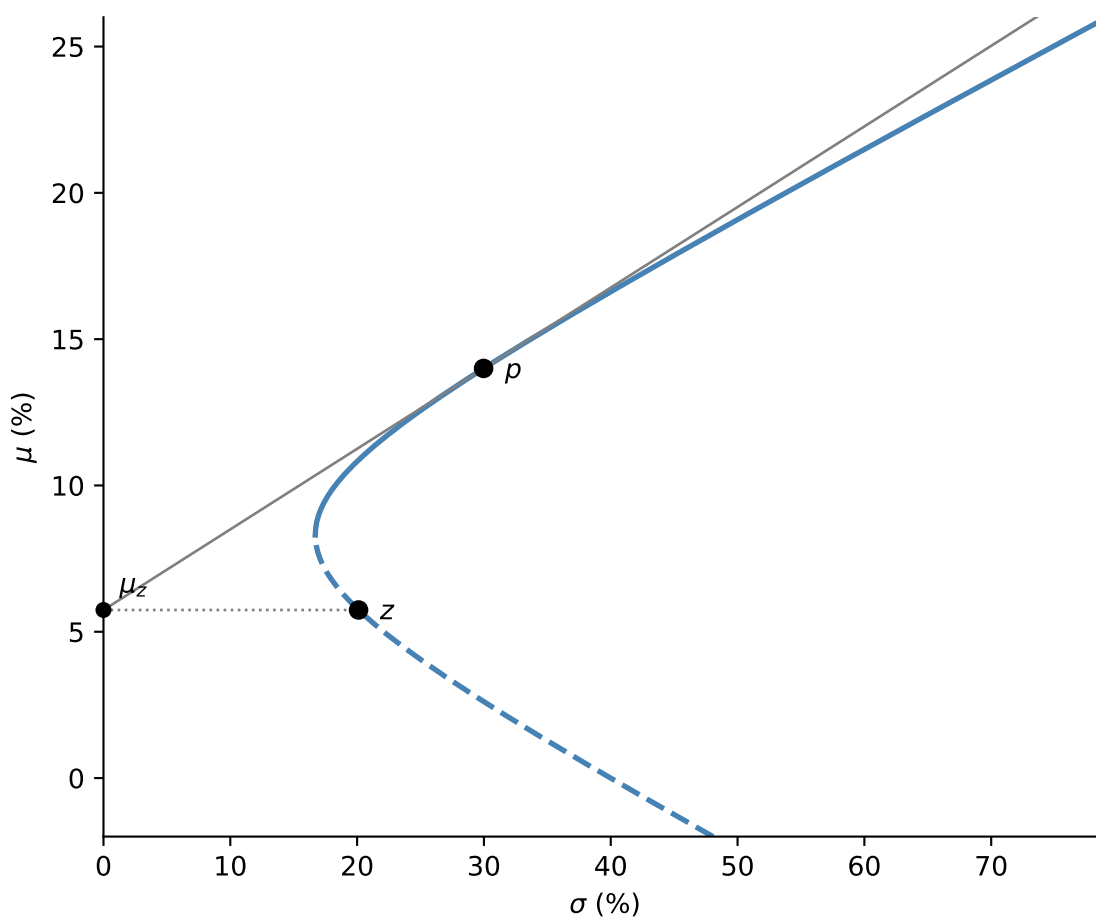


Figure 1: Zero-covariance portfolio. The tangent to the frontier at p meets the vertical axis at μ_z , and z is the frontier portfolio with that mean.

²Geometrically, the tangent at the minimum variance portfolio is vertical in (σ, μ) space, so it has no finite intercept with the μ -axis.

Property 3 (Beta Pricing with Frontier Portfolios). *Frontier portfolios contain all the information we need to price assets and carry all the systematic risk of the economy. Just pick any frontier portfolio p with return r_p , and compute its associated zero-covariance portfolio r_z . Then for any asset or portfolio i we have that*

$$r_i = (1 - \beta_i)r_z + \beta_i r_p + \varepsilon_i,$$

where

$$\beta_i = \frac{\text{Cov}(r_i, r_p)}{\sigma^2(r_p)}, \quad E(\varepsilon_i) = \text{Cov}(r_p, r_z) = \text{Cov}(r_p, \varepsilon_i) = \text{Cov}(r_z, \varepsilon_i) = 0.$$

Let $\mu_i = E(r_i)$. We start by re-writing (1) as

$$r_i = r_{p,i} + \varepsilon_i,$$

and note that $r_{p,i}$ is a frontier portfolio, and hence it can be represented as a portfolio of any two other frontier portfolios. Therefore, just pick an arbitrary frontier portfolio p (but different from the minimum variance portfolio)³ with expected return μ_p , and find its associated zero-covariance frontier portfolio z . We then form a portfolio composed of both frontier portfolios such that

$$r_{p,i} = (1 - \beta_i)r_z + \beta_i r_p,$$

with

$$\beta_i = \frac{\mu_i - \mu_z}{\mu_p - \mu_z}.$$

The choice for β_i guarantees that $E(r_{p,i}) = \mu_i$. Note that ε_i is uncorrelated with both r_z and r_p , and that r_z and r_p are also uncorrelated. Hence, the covariance of r_i and r_p is given by

$$\begin{aligned} \text{Cov}(r_i, r_p) &= \text{Cov}((1 - \beta_i)r_z + \beta_i r_p + \varepsilon_i, r_p) \\ &= (1 - \beta_i) \underbrace{\text{Cov}(r_z, r_p)}_0 + \beta_i \underbrace{\text{Cov}(r_p, r_p)}_{\sigma^2(r_p)} + \underbrace{\text{Cov}(\varepsilon_i, r_p)}_0 = \beta_i \sigma^2(r_p), \end{aligned}$$

³The restriction is necessary because, by Property 2, the zero-covariance portfolio z does not exist for the minimum variance portfolio.

which yields that

$$\beta_i = \frac{\text{Cov}(r_i, r_p)}{\sigma^2(r_p)}.$$

The result holds only because p is a frontier portfolio. Figure 2 computes betas for the three assets and for simulated portfolios of them against two benchmarks. Against the frontier portfolio p of Figure 1, every point lies exactly on the line through $(0, \mu_z)$ and $(1, \mu_p)$. Against asset 2, which lies inside the frontier, the same portfolios scatter and no line prices them. A linear relation between expected returns and betas therefore says only that the benchmark is a frontier portfolio, a point that returns in [Roll's critique](#) of the CAPM.

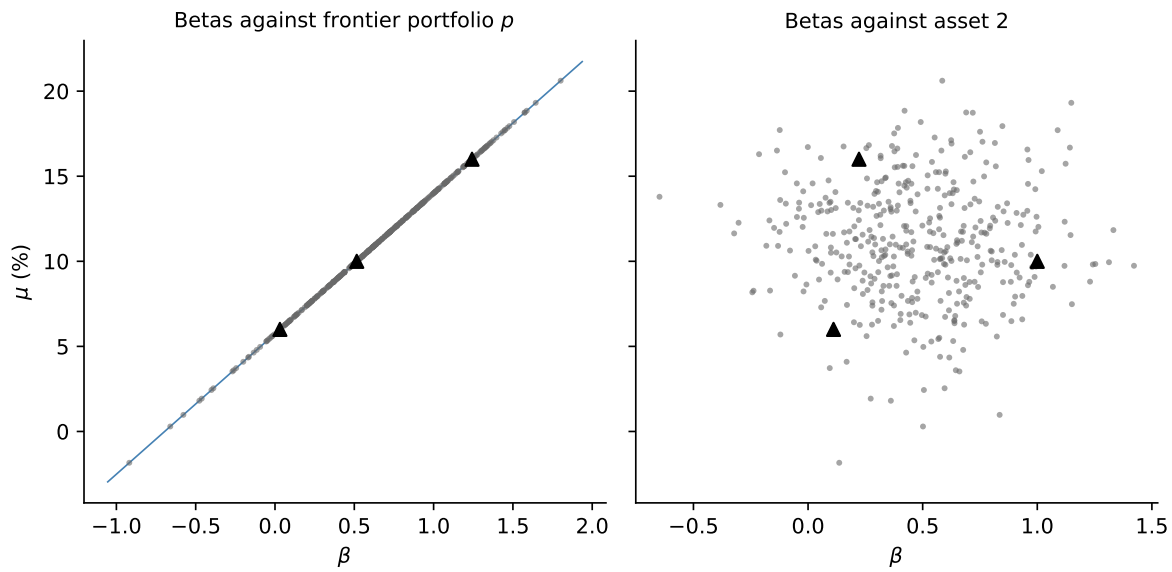


Figure 2: Expected returns against betas for the three assets (triangles) and simulated portfolios (dots). Left: betas against the frontier portfolio p , where all points lie on the line through $(0, \mu_z)$ and $(1, \mu_p)$. Right: betas against asset 2, which is not on the frontier.

References

Dybvig, Philip H., and Jonathan E. Ingersoll Jr. 1982. "Mean-Variance Theory in Complete Markets." *Journal of Business* 55 (2): 233–51.